STAT 3202: Practice 01

Autumn 2018, OSU

Exercise 1

Consider independent random variables X_1, X_2, and X_3 with

- $\text{E}[X_1] = 1$, $\text{Var}[X_1] = 4$
- $\text{E}[X_2] = 2$, $\text{SD}[X_1] = 3$
- $\text{E}[X_3] = 3$, $\text{SD}[X_1] = 5$

(a) Calculate $\text{E}[5X_1 + 2]$.

(b) Calculate $\text{E}[4X_1 + 2X_2 - 6X_3]$.

(c) Calculate $\text{Var}[5X_1 + 2]$.

(d) Calculate $\text{Var}[4X_1 + 2X_2 - 6X_3]$.

Exercise 2

Consider random variables H and Q with

- $\text{E}[H] = 3$, $\text{Var}[H] = 16$
- $\text{SD}[Q] = 4$, $\text{E}\left[\frac{Q^2}{5}\right] = 3.2$

(a) Calculate $\text{E}[5H^2 - 10]$.

(b) Calculate $\text{E}[Q]$.

Exercise 3

Consider a random variable S with probability density function

$$f(s) = \frac{1}{9000}(2s + 10), \quad 40 \leq s \leq 100.$$

(a) Calculate $\text{E}[S]$.

(b) Calculate $\text{SD}[S]$.

Exercise 4

Consider independent random variables X and Y with

- $X \sim N(\mu_X = 2, \sigma_X^2 = 9)$
- $Y \sim N(\mu_Y = 5, \sigma_Y^2 = 4)$

(a) Calculate $P[X > 5]$.

(b) Calculate $P[X + 2Y > 5]$.

Exercise 5

Consider random variables Y_1, Y_2, and Y_3 with

- $\text{E}[Y_1] = 1$, $\text{E}[Y_2] = -2$, $\text{E}[Y_3] = 3$
- $\text{Var}[Y_1] = 4$, $\text{Var}[Y_2] = 6$, $\text{Var}[Y_3] = 8$
- $\text{Cov}[Y_1, Y_2] = 1$, $\text{Cov}[Y_1, Y_3] = -1$, $\text{Cov}[Y_2, Y_3] = 0$

(a) Calculate $\text{Var}[3Y_1 - 2Y_2]$.

(b) Calculate $\text{Var}[3Y_1 - 4Y_2 + 2Y_3]$.

Exercise 6

Consider using $\hat{\xi}$ to estimate ξ.

(a) If $\text{Bias}[\hat{\xi}] = 5$ and $\text{Var}[\hat{\xi}] = 4$, calculate $\text{MSE}[\hat{\xi}]$

(b) If $\hat{\xi}$ is unbiased, $\xi = 6$, and $\text{MSE}[\hat{\xi}] = 30$, calculate $\text{E}\left[\hat{\xi}^2\right]$

Exercise 7

Using the identity

$$(\hat{\theta} - \theta) = \left(\hat{\theta} - \text{E}[\hat{\theta}]\right) + \left(\text{E}[\hat{\theta}] - \theta\right) = \left(\hat{\theta} - \text{E}[\hat{\theta}]\right) + \text{Bias}[\hat{\theta}]$$

show that

$$\text{MSE}[\hat{\theta}] = \text{E}\left[(\hat{\theta} - \theta)^2\right] = \text{Var}[\hat{\theta}] + \left(\text{Bias}[\hat{\theta}]\right)^2$$

Exercise 8

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a population with mean μ and variance σ^2.

Consider three estimators of μ:

$$\hat{\mu}_1 = \frac{X_1 + X_2 + X_3}{3}, \quad \hat{\mu}_2 = \frac{X_1}{4} + \frac{X_2 + \cdots + X_{n-1}}{2(n-2)} + \frac{X_n}{4}, \quad \hat{\mu}_3 = \bar{X},$$

Calculate the mean squared error for each estimator. (It will be useful to first calculate their bias and variances.)

Exercise 9

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution with density

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \theta \geq 0$$

Consider five estimators of θ:

$$\hat{\theta}_1 = X_1, \quad \hat{\theta}_2 = \frac{X_1 + X_2}{2}, \quad \hat{\theta}_3 = \frac{X_1 + 2X_2}{3}, \quad \hat{\theta}_4 = \bar{X}, \quad \hat{\theta}_5 = 5$$

Calculate the mean squared error for each estimator. (It will be useful to first calculate their bias and variances.)

Exercise 10

Suppose that $\mathrm{E}\left[\hat{\theta}_1\right] = \mathrm{E}\left[\hat{\theta}_2\right] = \theta$, $\mathrm{Var}\left[\hat{\theta}_1\right] = \sigma_1^2$, $\mathrm{Var}\left[\hat{\theta}_2\right] = \sigma_2^2$, and $\mathrm{Cov}\left[\hat{\theta}_1, \hat{\theta}_2\right] = \sigma_{12}$. Consider the unbiased estimator

$$\hat{\theta}_3 = a\hat{\theta}_1 + (1-a)\hat{\theta}_2.$$

If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, what value should be chosen for the constant a in order to minimize the variance and thus mean squared error of $\hat{\theta}_3$ as an estimator of θ?

Exercise 11

Let Y have a binomial distribution with parameters n and p. Consider two estimators for p:

$$\hat{p}_1 = \frac{Y}{n}$$

and

$$\hat{p}_2 = \frac{Y + 1}{n + 2}$$

For what values of p does $\hat{p}_2$ achieve a lower mean square error than $\hat{p}_1$?

Exercise 12

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a population with mean μ and variance σ^2.

Create an unbiased estimator for μ^2. Hint: Start with $\bar{X}^2$.

Exercise 13

Let $X_1, X_2, X_3, \ldots, X_n$ be iid random variables form U$(\theta, \theta + 2)$. (That is, a uniform distribution with a minimum of θ and a maximum of $\theta + 2$.)

Consider the estimator

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

(a) Calculate the **bias** of $\hat{\theta}$ when estimating θ.

(b) Calculate the **variance** of $\hat{\theta}$.

(c) Calculate the **mean squared error** of $\hat{\theta}$ when estimating θ.
