STAT 3202: Practice 03

Autumn 2018, OSU

Exercise 1

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

(a) Obtain a method of moments **estimator** for λ, $\tilde{\lambda}$. Calculate an **estimate** using this *estimator* when

$$x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 2.$$

Solution:

Recall that for a Poisson distribution we have $\text{E}[X] = \lambda$.

Now to obtain the method of moments estimator we simply equate the first population mean to the first sample mean. (And then we need to "solve" this equation for λ...)

$$\text{E}[X] = \bar{X}\lambda = \bar{X}$$

Thus, after "solving" we obtain the method of moments *estimator*.

$$\boxed{\tilde{\lambda} = \bar{X}}$$

Thus for the given data we can use this estimator to calculate the *estimate*.

$$\tilde{\lambda} = \bar{x} = \frac{1}{4}(1 + 2 + 4 + 2) = \boxed{2.25}$$

(b) Find the maximum likelihood **estimator** for λ, $\hat{\lambda}$. Calculate an **estimate** using this *estimator* when

$$x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 2.$$

Solution:

$$L(\lambda) = \prod_{i=1}^{n} f(x_i \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} (x_i!)}$$

$$\log L(\lambda) = \left(\sum_{i=1}^{n} x_i\right) \log \lambda - n\lambda - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{d}{d\lambda} \log L(\lambda) = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0$$

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\frac{d^2}{d\lambda^2} \log L(\lambda) = -\frac{\sum_{i=1}^{n} x_i}{\lambda^2} < 0$$

We then have the *estimator*, and for the given data, the *estimate*.

$$\boxed{\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i} = \frac{1}{4}(1 + 2 + 4 + 2) = \boxed{2.25}$$

(c) Find the maximum likelihood **estimator** of $P[X = 4]$, call it $\hat{P}[X = 4]$. Calculate an **estimate** using this *estimator* when

$$x_1 = 1, \ x_2 = 2, \ x_3 = 4, \ x_4 = 2.$$

Solution:

Here we use the invariance property of the MLE. Since $\hat{\lambda}$ is the MLE for λ then

$$\boxed{\hat{P}[X = 4] = \frac{\hat{\lambda}^4 e^{-\hat{\lambda}}}{4!}}$$

is the maximum *likelihood estimator* for $P[X = 4]$.

For the given data we can calculate an *estimate* using this estimator.

$$\hat{P}[X = 4] = \frac{\hat{\lambda}^4 e^{-\hat{\lambda}}}{4!} = \frac{2.25^4 e^{-2.25}}{4!} = \boxed{0.1126}$$

Exercise 2

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$.

Find a method of moments **estimator** for the *parameter vector* $\left(\theta, \sigma^2\right)$.

Solution:

Since we are estimating two parameters, we will need two population and sample moments.

$$\text{E}[X] = \theta$$

$$\text{E}\left[X^2\right] = \text{Var}\left[X\right] + \left(\text{E}[X]\right)^2 = \sigma^2 + \theta^2$$

We equate the first population moment to the first sample moment, $\bar{x}$ and we equate the second population moment to the second sample moment, $\overline{X^2} = \frac{1}{n} \sum_{i=1}^{n} X_i^2$.

$$\mathrm{E}[X] = \bar{X}$$

$$\mathrm{E}\left[X^2\right] = \overline{X^2}$$

For this example, that is,

$$\theta = \bar{X}$$

$$\sigma^2 + \theta^2 = \overline{X^2}$$

Solving this system of equations for θ and σ^2 we find the method of moments estimators.

$$\boxed{\begin{aligned} &\tilde{\theta} = \bar{X} \\ &\tilde{\sigma}^2 = \overline{X^2} - (\bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \end{aligned}}$$

Exercise 3

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(1, \sigma^2)$.

Find a method of moments **estimator** of σ^2, call it $\tilde{\sigma}^2$.

Solution:

The first moment is not useful because it is not a function of the parameter of interest σ^2.

$$\mathrm{E}[X] = 1$$

As a results, we instead use the second moment

$$\mathrm{E}\left[X^2\right] = \mathrm{Var}[X] + (\mathrm{E}[X])^2 = \sigma^2 + 1^2 = \sigma^2 + 1$$

We equate this second population moment to the second population moment, $\overline{X^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

$$\mathrm{E}\left[X^2\right] = \overline{X^2}$$

$$\sigma^2 + 1 = \overline{X^2}$$

Now solving for σ^2 we obtain the method of moments estimator.

$$\boxed{\tilde{\sigma}^2 = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) - 1}$$

Exercise 4

Let $X_1, X_2, \ldots, X_n$ iid from a population with pdf

$$f(x \mid \theta) = \frac{1}{\theta} x^{(1-\theta)/\theta}, \quad 0 < x < 1,\ 0 < \theta < \infty$$

(a) Find the maximum likelihood **estimator** of θ, call it $\hat{\theta}$. Calculate an **estimate** using this *estimator* when

$$x_1 = 0.10,\ x_2 = 0.22,\ x_3 = 0.54,\ x_4 = 0.36.$$

Solution:

$$L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} x_i^{(1-\theta)/\theta} = \theta^{-n} \left(\prod_{i=1}^{n} x_i \right)^{\frac{1-\theta}{\theta}}$$

$$\log L(\theta) = -n \log \theta + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \log x_i = -n \log \theta + \frac{1}{\theta} \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \log x_i$$

$$\frac{d}{d\theta} \log L(\theta) = -\frac{n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \log x_i = 0$$

$$\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \log x_i$$

Note that $\hat{\theta} > 0$, since each $\log x_i < 0$ since $0 < x_i < 1$.

$$\frac{d^2}{d\theta^2} \log L(\theta) = \frac{n}{\theta^2} + \frac{2}{\theta^3} \sum_{i=1}^{n} \log x_i$$

$$\frac{d^2}{d\theta^2} \log L(\hat{\theta}) = \frac{n}{\hat{\theta}^2} + \frac{2}{\hat{\theta}^3} \left(-n\hat{\theta} \right) = \frac{n}{\hat{\theta}^2} - \frac{2n}{\hat{\theta}^2} = -\frac{n}{\hat{\theta}^2} < 0$$

We then have the *estimator*, and for the given data, the *estimate*.

$$\boxed{\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \log x_i} = -\frac{1}{4} \log(0.10 \cdot 0.22 \cdot 0.54 \cdot 0.36) = \boxed{1.3636}$$

(b) Obtain a method of moments **estimator** for θ, $\tilde{\theta}$. Calculate an **estimate** using this *estimator* when

$$x_1 = 0.10,\ x_2 = 0.22,\ x_3 = 0.54,\ x_4 = 0.36.$$

Solution:

$$\mathrm{E}[X] = \int_0^1 x \cdot \frac{1}{\theta} x^{(1-\theta)/\theta} dx = \text{... some calculus happens...} = \frac{1}{\theta + 1}$$

$$\mathrm{E}[X] = \bar{X}$$

$$\frac{1}{\theta + 1} = \bar{X}$$

Solving for θ results in the method of moments *estimator*.

$$\boxed{\tilde{\theta} = \frac{1 - \bar{X}}{\bar{X}}}$$

$$\bar{x} = \frac{1}{4}(0.10 + 0.22 + 0.54 + 0.36) = 0.305$$

Thus for the given data we can calculate the *estimate*.

$$\tilde{\theta} = \frac{1 - \bar{x}}{\bar{x}} = \frac{1 - 0.305}{0.305} = \boxed{2.2787}$$

Exercise 5

Let $X_1, X_2, \ldots, X_n$ iid from a population with pdf

$$f(x \mid \theta) = \frac{\theta}{x^2}, \quad 0 < \theta \leq x$$

Obtain the maximum likelihood **estimator** for θ, $\hat{\theta}$.

Solution:

First, be aware that the values of x for this pdf are restricted by the value of θ.

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^2} \quad 0 < \theta \leq x_i \text{ for all } x_i$$

$$= \frac{\theta^n}{\prod_{i=1}^{n} x_i^2} \quad 0 < \theta \leq \min\{x_i\}$$

$$\log L(\theta) = n \log \theta - 2 \sum_{i=1}^{n} \log x_i$$

$$\frac{d}{d\theta} \log L(\theta) = \frac{n}{\theta} > 0$$

So, here we have a log-likelihood that is increasing in regions where it is not zero, that is, when $\theta \min\{x_i\}$. Thus, the likelihood is the largest allowable value of θ in this region, thus the maximum likelihood estimator is given by

$$\boxed{\hat{\theta} = \min\{X_i\}}$$

Exercise 6

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x, \alpha) = \alpha^{-2} x e^{-x/\alpha}, \quad x > 0, \ \alpha > 0$$

(a) Obtain the maximum likelihood **estimator** of α, $\hat{\alpha}$. Calculate the **estimate** when

$$x_1 = 0.25, \ x_2 = 0.75, \ x_3 = 1.50, \ x_4 = 2.5, \ x_5 = 2.0.$$

Solution:

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\alpha) = \prod_{i=1}^{n} f(x_i; \alpha) = \prod_{i=1}^{n} \alpha^{-2} x_i e^{-x_i/\alpha} = \alpha^{-2n} \left(\prod_{i=1}^{n} x_i \right) \exp\left(\frac{-\sum_{i=i}^{n} x_i}{\alpha} \right)$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\alpha) = -2n \log \alpha + \sum_{i=i}^{n} \log x_i - \frac{\sum_{i=i}^{n} x_i}{\alpha}$$

To maximize this function, we take a **derivative** with respect to α.

$$\frac{d}{d\alpha} \log L(\alpha) = \frac{-2n}{\alpha} + \frac{\sum_{i=i}^{n} x_i}{\alpha^2}$$

We set this derivative equal to **zero**, then **solve** for α.

$$\frac{-2n}{\alpha} + \frac{\sum_{i=i}^{n} x_i}{\alpha^2} = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\boxed{\hat{\alpha} = \frac{\sum_{i=i}^{n} x_i}{2n} = \frac{\bar{x}}{2}}$$

Using the given data, we obtain an *estimate*.

$$\hat{\alpha} = \frac{0.25 + 0.75 + 1.50 + 2.50 + 2.0}{2 \cdot 5} = \boxed{0.70}$$

(We should also verify that this point is a maxmimum, which is omitted here.)

(b) Obtain the method of moments **estimator** of α, $\tilde{\alpha}$. Calculate the **estimate** when

$$x_1 = 0.25, \ x_2 = 0.75, \ x_3 = 1.50, \ x_4 = 2.5, \ x_5 = 2.0.$$

Hint: Recall the probability density function of an exponential random variable.

$$f(x \mid \theta) = \frac{1}{\theta}e^{-x/\theta}, \quad x > 0, \ \theta > 0$$

Note that, the moments of this distribution are given by

$$E[X^k] = \int_0^\infty \frac{x^k}{\theta}e^{-x/\theta} = k! \cdot \theta^k.$$

This hint will also be useful in the next exercise.

Solution:

We first obtain the first **population moment**. Notice the integration is done by identifying the form of the integral is that of the second moment of an exponential distribution.

$$\mathrm{E}[X] = \int_0^\infty x \cdot \alpha^{-2}xe^{-x/\alpha}dx = \frac{1}{\alpha}\int_0^\infty \frac{x^2}{\alpha}e^{-x/\alpha}dx = \frac{1}{\alpha}(2\alpha^2) = 2\alpha$$

We then set the first population moment, which is a function of α, equal to the first **sample moment**.

$$2\alpha = \frac{\sum_{i=i}^n x_i}{n}$$

Solving for α, we obtain the method of moments *estimator*.

$$\boxed{\tilde{\alpha} = \frac{\sum_{i=i}^n x_i}{2n} = \frac{\bar{x}}{2}}$$

Using the given data, we obtain an *estimate*.

$$\tilde{\alpha} = \frac{0.25 + 0.75 + 1.50 + 2.50 + 2.0}{2 \cdot 5} = \boxed{0.70}$$

Note that, in this case, the MLE and MoM estimators are the same.

Exercise 7

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a distribution with probability density function

$$f(x \mid \beta) = \frac{1}{2\beta^3}x^2e^{-x/\beta}, \quad x > 0, \ \beta > 0$$

(a) Obtain the maximum likelihood **estimator** of β, $\hat{\beta}$. Calculate the **estimate** when

$$x_1 = 2.00, \ x_2 = 4.00, \ x_3 = 7.50, \ x_4 = 3.00.$$

Solution:

We first obtain the likelihood by **multiplying** the probability density function for each X_i. We then **simplify** this expression.

$$L(\beta) = \prod_{i=1}^{n} f(x_i; \beta) = \prod_{i=1}^{n} \frac{1}{2\beta^3} x^2 e^{-x/\beta} = 2^{-n} \beta^{-3n} \left(\prod_{i=1}^{n} x_i^2 \right) \exp \left(\frac{-\sum_{i=i}^{n} x_i}{\beta} \right)$$

Instead of directly maximizing the likelihood, we instead maximize the **log**-likelihood.

$$\log L(\beta) = -n \log 2 - 3n \log \beta + \sum_{i=i}^{n} \log x_i^2 - \frac{\sum_{i=i}^{n} x_i}{\beta}$$

To maximize this function, we take a **derivative** with respect to β.

$$\frac{d}{d\beta} \log L(\beta) = \frac{-3n}{\beta} + \frac{\sum_{i=i}^{n} x_i}{\beta^2}$$

We set this derivative equal to **zero**, then **solve** for β.

$$\frac{-3n}{\beta} + \frac{\sum_{i=i}^{n} x_i}{\beta^2} = 0$$

Solving gives our *estimator*, which we denote with a **hat**.

$$\boxed{\hat{\beta} = \frac{\sum_{i=i}^{n} x_i}{3n} = \frac{\bar{x}}{3}}$$

Using the given data, we obtain an *estimate*.

$$\hat{\beta} = \frac{2.00 + 4.00 + 7.50 + 3.00}{3 \cdot 4} = \boxed{1.375}$$

(We should also verify that this point is a maxmimum, which is omitted here.)

(b) Obtain the method of moments **estimator** of β, $\tilde{\beta}$. Calculate the **estimate** when

$$x_1 = 2.00, \ x_2 = 4.00, \ x_3 = 7.50, \ x_4 = 3.00.$$

Solution:

We first obtain the first **population moment**. Notice the integration is done by identifying the form of the integral is that of the third moment of an exponential distribution.

$$\mathrm{E}[X] = \int_0^\infty x \cdot \frac{1}{2\beta^3} x^2 e^{-x/\beta} dx = \frac{1}{2\beta^2} \int_0^\infty \frac{x^3}{\beta} e^{-x/\beta} dx = \frac{1}{2\beta^2} (6\beta^3) = 3\beta$$

We then set the first population moment, which is a function of β, equal to the first **sample moment**.

$$\mathrm{E}[X] = \bar{X}$$

$$3\beta = \frac{\sum_{i=i}^{n} x_i}{n}$$

Solving for β, we obtain the method of moments *estimator*.

$$\boxed{\tilde{\beta} = \frac{\sum_{i=i}^{n} x_i}{3n} = \frac{\bar{x}}{3}}$$

Using the given data, we obtain an *estimate.*

$$\tilde{\beta} = \frac{2.00 + 4.00 + 7.50 + 3.00}{3 \cdot 4} = \boxed{1.375}$$

Note again, the MLE and MoM estimators are the same.
