# Estimation II: Consistency

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# The Usual Setup

Suppose we are interested in the value of some parameter $\theta$ that describes a feature of a population. We draw a random sample from the population, $X_1, \ldots, X_n$, and have an estimator $\hat{\theta}$ which is a function of the sample: $\hat{\theta} = \hat{\theta}(X_1, \ldots, X_n)$.

- *Idea:* We'd like $\hat{\theta}$ to get "closer" and closer to $\theta$ as we draw larger and larger samples.

# Definition: Statistical Consistency

An estimator $\hat{\theta}_n$ is said to be a **consistent estimator** of $\theta$ if, for any positive $\epsilon$,

$$\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| \leq \epsilon) = 1$$

or, equivalently,

$$\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| > \epsilon) = 0$$

We say that $\hat{\theta}_n$ **converges in probability** to $\theta$ and we write $\hat{\theta}_n \xrightarrow{P} \theta$.

# Example: Using the Definition

An estimator $\hat{\theta}_n$ is said to be a **consistent estimator** of $\theta$ if, for any positive $\epsilon$,

$$\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| \leq \epsilon) = 1$$

- Let $X_1, X_2, \ldots, X_n$ be iid $N(\theta, 1)$ and consider $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Use the definition of consistency to show that $\bar{X}_n$ is a consistent estimator of $\theta$.

# An Easier Method

**Theorem:** An *unbiased* estimator $\hat{\theta}_n$ for $\theta$ is a consistent estimator of $\theta$ if

$$\lim_{n \to \infty} \text{Var}\left[\hat{\theta}_n\right] = 0$$

- Proof?

# Example: This is Easier

**Theorem:** An *unbiased* estimator $\hat{\theta}_n$ for $\theta$ is a consistent estimator of $\theta$ if

$$\lim_{n \to \infty} \text{Var}\left[\hat{\theta}_n\right] = 0$$

- Again letting $X_1, X_2, \ldots, X_n$ be iid $N(\theta, 1)$ and consider $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Show that $\bar{X}_n$ is a consistent estimator of $\theta$.

# Example

Suppose that $X_1, X_2, \ldots, X_n$ are an iid sample from the distribution

$$f(x;\theta) = \frac{1}{2}(1+\theta x), \quad -1 < x < 1, -1 < \theta < 1.$$

Previously:

- $\hat{\theta} = 3\bar{X}_n$ is an unbiased estimator of $\theta$.

Is $3\bar{X}_n$ a consistent estimator of $\theta$?

# The (Weak) Law of Large Numbers

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample such that

- $\text{E}[Y_i] = \mu$
- $\text{Var}[Y_i] = \sigma^2$.

Show that $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is a consistent estimator of $\mu$. Thus, show that

$$\bar{Y}_n \xrightarrow{P} \mu$$

# Additional Results

**Theorem:** Suppose that $\hat{\theta}_n \xrightarrow{P} \theta$ and that $\hat{\beta}_n \xrightarrow{P} \beta$.

- $\hat{\theta}_n + \hat{\beta}_n \xrightarrow{P} \theta + \beta$
- $\hat{\theta}_n \times \hat{\beta}_n \xrightarrow{P} \theta \times \beta$
- $\hat{\theta}_n \div \hat{\beta}_n \xrightarrow{P} \theta \div \beta$
- If $g(\cdot)$ is a real valued function that is continuous at $\theta$, then $g(\hat{\theta}_n) \xrightarrow{P} g(\theta)$

# Example

**Theorem:** Suppose that $\hat{\theta}_n \xrightarrow{P} \theta$ and that $\hat{\beta}_n \xrightarrow{P} \beta$.

- $\hat{\theta}_n + \hat{\beta}_n \xrightarrow{P} \theta + \beta$
- $\hat{\theta}_n \times \hat{\beta}_n \xrightarrow{P} \theta \times \beta$
- $\hat{\theta}_n \div \hat{\beta}_n \xrightarrow{P} \theta \div \beta$
- If $g(\cdot)$ is a real valued function that is continuous at $\theta$, then $g(\hat{\theta}_n) \xrightarrow{P} g(\theta)$

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample such that

- $\mathrm{E}[Y_i] = \mu$
- $\mathrm{Var}[Y_i] = \sigma^2$.

Suggest a consistent estimator for $\mu^2$.

# Example

**Theorem:** Suppose that $\hat{\theta}_n \xrightarrow{P} \theta$ and that $\hat{\beta}_n \xrightarrow{P} \beta$.

- $\hat{\theta}_n + \hat{\beta}_n \xrightarrow{P} \theta + \beta$
- $\hat{\theta}_n \times \hat{\beta}_n \xrightarrow{P} \theta \times \beta$
- $\hat{\theta}_n \div \hat{\beta}_n \xrightarrow{P} \theta \div \beta$
- If $g(\cdot)$ is a real valued function that is continuous at $\theta$, then $g(\hat{\theta}_n) \xrightarrow{P} g(\theta)$

Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu_X, \sigma_X^2)$. Also, let $Y_1, Y_2, \ldots, Y_m$ be iid $N(\mu_Y, \sigma_Y^2)$.

Suggest a consistent estimator for $\mu_X - \mu_Y$.