Estimation III: Method of Moments and Maximum Likelihood

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Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

How should we estimate $\lambda$?
The $k^{th}$ **population moment** of a RV (about the origin) is

$$
\mu'_k = E[Y^k]
$$

The $k^{th}$ **sample moment** is

$$
m'_k = \overline{Y^k} = \frac{1}{n} \sum_{i=1}^{n} Y_i^k
$$
The Method of Moments (MoM) consists of equating sample moments and population moments. If a population has \( t \) parameters, the MOM consists of solving the system of equations

\[
m'_k = \mu'_k, \quad k = 1, 2, \ldots, t
\]

for the \( t \) parameters.
Example: Poisson

Let $X_1, X_2, \ldots, X_n \sim iid \text{ Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

Find a method of moments estimator of $\lambda$, call it $\tilde{\lambda}$. 
Let $X_1, X_2, \ldots, X_n$ be iid $N(\theta, \sigma^2)$.

Use the method of moments to estimate the parameter vector $(\theta, \sigma^2)$. 
Example: Normal, Mean Known

Let $X_1, X_2, \ldots, X_n$ be iid $N(1, \sigma^2)$.

Find a method of moments estimator of $\sigma^2$, call it $\bar{\sigma}^2$. 
Calculus???
A Game Show / An Idea

Box 1

Box 2

Box 3
Let $Y \sim \text{binom}(n = 100, p)$.

Suppose we observe a single observation $x = 60$. 
Log Rules

- $x^m x^n = x^{m+n}$
- $(x^m)^n = x^{mn}$
- $\log(ab) = \log(a) + \log(b)$
- $\log(a/b) = \log(a) - \log(b)$
- $\log(a^b) = b \log(a)$
- $\prod_{i=1}^n x_i = x_1 \cdot x_2 \cdots \cdot x_n$
- $\prod_{i=1}^n x_i^a = (\prod_{i=1}^n x_i)^a$
- $\log \left( \prod_{i=1}^n x_i \right) = \sum_{i=1}^n \log(x_i)$
Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

Find the maximum likelihood estimator of $\lambda$, call it $\hat{\lambda}$. 
Example: Poisson

Let $X_1, X_2, \ldots, X_n \sim iid \text{ Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

Calculate the maximum likelihood estimate of $\lambda$, when

$$x_1 = 1, \ x_2 = 2, \ x_3 = 4, \ x_4 = 2.$$
Given a random sample $X_1, X_2, \ldots, X_n$ from a population with parameter $\theta$ and density or mass $f(x \mid \theta)$, we have:

The Likelihood, $L(\theta)$,

$$L(\theta) = f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i \mid \theta)$$

The Maximum Likelihood Estimator, $\hat{\theta}$

$$\hat{\theta} = \arg\max_{\theta} L(\theta) = \arg\max_{\theta} \log L(\theta)$$
Invariance Principle

If $\hat{\theta}$ is the MLE of $\theta$ and the function $h(\theta)$ is continuous, then $h(\hat{\theta})$ is the MLE of $h(\theta)$.

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

- **Example:** Find the maximum likelihood estimator of $P[X = 4]$, call it $\hat{P}[X = 4]$. Calculate an estimate using this estimator when

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = 2.$$
Some Brief History
Who Is This?
Let $X_1, X_2, \ldots, X_n$ iid from a population with pdf

$$f(x \mid \theta) = \frac{1}{\theta} x^{(1-\theta)/\theta}, \quad 0 < x < 1, \quad 0 < \theta < \infty$$

Find the maximum likelihood estimator of $\theta$, call it $\hat{\theta}$. 
Let $X_1, X_2, \ldots, X_n$ iid from a population with pdf

$$f(x \mid \theta) = \frac{\theta}{x^2}, \quad 0 < \theta \leq x < \infty$$

Find the maximum likelihood estimator of $\theta$, call it $\hat{\theta}$. 
Example: Gamma

Let $X_1, X_2, \ldots, X_n \sim \text{iid gamma}(\alpha, \beta)$ with $\alpha$ known.

Find the maximum likelihood estimator of $\beta$, call it $\hat{\beta}$.
Next Time

- More examples?
- Why does this work?
- Why do we need both MLE and MoM?
- How do we use these methods in practice?