1. Let $\tau > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\tau) = \frac{\tau^5}{8} x^{14} e^{-\tau x^3}, \qquad x > 0.$$

Obtain the maximum likelihood estimator of τ, $\hat{\tau}$.

$$L(\tau) = \prod_{i=1}^{n}\left(\frac{\tau^5}{8} x_i^{14} e^{-\tau x_i^3}\right) = \frac{\tau^{5n}}{8^n}\left(\prod_{i=1}^{n} x_i^{14}\right) e^{-\tau \sum_{i=1}^{n} x_i^3}$$

$$\ln L(\tau) = 5n \cdot \ln\tau - n \cdot \ln 8 + 14 \sum_{i=1}^{n} \ln x_i - \tau \cdot \sum_{i=1}^{n} x_i^3$$

$$(\ln L(\tau))' = \frac{5n}{\tau} - \sum_{i=1}^{n} x_i^3 = 0$$

$$\Rightarrow \qquad \hat{\tau} = \frac{5n}{\sum_{i=1}^{n} X_i^3}.$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = \frac{2(\theta - x)}{\theta^2} \qquad 0 < x < \theta \qquad \theta > 0.$$

The method of moments estimator of θ, $\tilde{\theta}$, is given by

$$\tilde{\theta} = 3 \cdot \overline{X} = 3 \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} X_i$$

a) Is $\tilde{\theta}$ an unbiased estimator for θ?

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_0^{\theta} x \cdot \left(\frac{2}{\theta} - \frac{2}{\theta^2} x \right) dx = \left(\frac{x^2}{\theta} - \frac{2}{3} \cdot \frac{x^3}{\theta^2} \right) \Bigg|_0^{\theta} = \frac{\theta}{3}.$$

$$E(\tilde{\theta}) = E(3\,\overline{X}) = 3\,E(\overline{X}) = 3\,\mu = 3\,\frac{\theta}{3} = \theta.$$

$\Rightarrow$ $\tilde{\theta}$ an unbiased estimator for θ.

b) Find $\text{Var}(\tilde{\theta})$.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_0^{\theta} x^2 \cdot \left(\frac{2}{\theta} - \frac{2}{\theta^2} x \right) dx = \frac{\theta^2}{6}.$$

$$\sigma^2 = \text{Var}(X) = \frac{\theta^2}{6} - \frac{\theta^2}{9} = \frac{\theta^2}{18}.$$

$$\text{Var}(\tilde{\theta}) = \text{Var}(3\,\overline{X}) = 9\,\text{Var}(\overline{X}) = 9 \cdot \frac{\sigma^2}{n} = 9 \cdot \frac{\theta^2}{18\,n} = \frac{\theta^2}{2\,n}.$$

3. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\lambda) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}}\, e^{-\lambda x^2}, \qquad x > 0, \qquad \lambda > 0.$$

a) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n} \left(\frac{2\sqrt{\lambda}}{\sqrt{\pi}}\, e^{-\lambda x_i^2} \right).$$

$$\ln L(\lambda) = n \cdot \ln 2 + \frac{n}{2} \cdot \ln\lambda - \frac{n}{2} \cdot \ln\pi - \lambda \cdot \sum_{i=1}^{n} x_i^2 .$$

$$(\ln L(\lambda))' = \frac{n}{2\lambda} - \sum_{i=1}^{n} x_i^2 = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{n}{2\sum_{i=1}^{n} X_i^2}.$$

b) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find the maximum likelihood estimate of λ.

$x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.

$$\sum_{i=1}^{n} x_i^2 = 4.5. \qquad \hat{\lambda} = \mathbf{\frac{4}{9}} \approx 0.444.$$

4. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\lambda) = \frac{\lambda^2}{2} e^{-\lambda\sqrt{x}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Find the maximum likelihood estimator of λ, $\hat{\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda^2}{2} e^{-\lambda\sqrt{x_i}} \right) = \frac{\lambda^{2n}}{2^n} e^{-\lambda\Sigma\sqrt{x_i}}.$$

$$\ln L(\lambda) = 2n \cdot \ln\lambda - n \cdot \ln 2 - \lambda \cdot \sum_{i=1}^{n} \sqrt{x_i}.$$

$$(\ln L(\lambda))' = \frac{2n}{\lambda} - \sum_{i=1}^{n} \sqrt{x_i} = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} \sqrt{X_i}}.$$

b) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$.
Find the maximum likelihood estimate of λ, $\hat{\lambda}$.

$$\sum_{i=1}^{n} \sqrt{x_i} = 0.9 + 1.4 + 0.6 + 0.3 = 3.2.$$

$$\hat{\lambda} = \frac{2 \cdot 4}{3.2} = \mathbf{2.5}.$$

5. Several employees of *Al's Building Construction* complain that the variance of the employees' monthly salary amounts is too large. In a random sample of 10 employees, the sample standard deviation of the monthly salary amounts was $210.

a) Use $\alpha = 0.05$ to test $H_0 : \sigma \le 150$ vs. $H_1 : \sigma > 150$. Report the value of the test statistic, the critical value(s), and state your decision.

$s = 210.$ $n = 10.$

Test Statistic: $\chi^2 = \frac{(n-1)\cdot s^2}{\sigma_0^2} = \frac{(10-1)\cdot 210^2}{150^2} = \mathbf{17.64}.$

Rejection Region: Right – tailed.

Reject H_0 if $\chi^2 > \chi^2_\alpha$ $n - 1 = 9$ degrees of freedom.

$\alpha = 0.05$ $\chi^2_{0.05} = \mathbf{16.92}.$

Reject H_0 if $\chi^2 > 16.92.$

The value of the test statistic **does** fall into the Rejection Region.

Reject H_0 at $\alpha = 0.05$.

b) Using the chi-square distribution table only, what is the p-value of the test in part (a)? (You may give a range.)

$19.02 > 17.64 > 16.92$ **0.025 < p-value < 0.05**.

6. The service time in queues should not have a large variance; otherwise, the queue tends to build up. A bank regularly checks the service times of its tellers to determine their variance. A random sample of 22 service times (in minutes) gives $s^2 = 8$. Assume the service times are normally distributed.

a) Find a 95% confidence interval for the overall variance of service time at the bank.

$s^2 = 8.$ $n = 22.$

The confidence interval : $$\left(\frac{(n-1)\cdot s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2}} \right).$$

95% confidence level $\alpha = 0.05$ $\alpha/2 = \mathbf{0.025}.$

$n - 1 = \mathbf{21}$ degrees of freedom.

$$\chi^2_{\alpha/2} = \chi^2_{0.025} = 35.48. \qquad \chi^2_{1-\alpha/2} = \chi^2_{0.975} = 10.28.$$

$$\left(\frac{(22-1)\cdot 8}{35.48}, \frac{(22-1)\cdot 8}{10.28} \right)$$

(4.735 , 16.34)

b) Find the two one-sided 95% confidence intervals for the overall variance of service time at the bank.

$$\left(\frac{(n-1)\cdot s^2}{\chi^2_{\alpha}}, \infty \right) \qquad \left(0, \frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha}} \right)$$

$$\chi^2_{\alpha} = \chi^2_{0.05} = 32.67. \qquad \chi^2_{1-\alpha} = \chi^2_{0.95} = 11.59.$$

$$\left(\frac{(22-1)\cdot 8}{32.67}, \infty \right) \qquad \left(0, \frac{(22-1)\cdot 8}{11.59} \right)$$

(5.14 , ∞) **(0 , 14.5)**

7. In a random sample of 200 students who took Exam P in 2017, 74 successfully passed the exam. (Exam P [Probability] is the first Actuarial Science exam.) Suppose also that in a random sample of 150 students who took Exam FM in 2017, 45 successfully passed the exam. (Exam FM [Financial Mathematics] is the second Actuarial Science exam.)

a) Construct a 95% confidence interval for the difference between the overall proportions of students who passed Exam P and Exam FM in 2017.

$\hat{p}_{\text{P}} = 0.37.$ $\qquad\qquad\qquad \hat{p}_{\text{FM}} = \dfrac{45}{150} = 0.30.$

$$\left(\hat{p}_{\text{P}} - \hat{p}_{\text{FM}}\right) \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_{\text{P}} \cdot \left(1 - \hat{p}_{\text{P}}\right)}{n_{\text{P}}} + \frac{\hat{p}_{\text{FM}} \cdot \left(1 - \hat{p}_{\text{FM}}\right)}{n_{\text{FM}}}}$$

95% confidence level, $\qquad z_{\alpha/2} = z_{0.025} = 1.96.$

$$\left(0.37 - 0.30\right) \pm 1.96 \cdot \sqrt{\frac{0.37 \cdot 0.63}{200} + \frac{0.30 \cdot 0.70}{150}}$$

$\mathbf{0.070 \pm 0.099}$ $\qquad\qquad\qquad \mathbf{(-0.029\,,\,0.169)}$

b) At a 10% level of significance, test whether Exam P had a higher success rate than Exam FM in 2017. Find the p-value of this test.

$H_0 : p_{\text{P}} = p_{\text{FM}}$ vs. $H_1 : p_{\text{P}} > p_{\text{FM}}.$ $\qquad \hat{p} = \dfrac{74 + 45}{200 + 150} = \dfrac{119}{350} = 0.34.$

Test Statistic: $\qquad Z = \dfrac{0.37 - 0.30}{\sqrt{0.34 \cdot 0.66 \cdot \left(\dfrac{1}{200} + \dfrac{1}{150}\right)}} = \mathbf{1.37}.$

P-value: $\qquad$ Right – tailed.

P-value = (area of the right tail) = $P(Z \geq 1.37)$ = **0.0853**.

8. A random sample of 9 adult white rhinos had the sample mean weight of 5,100 pounds and the sample standard deviation of 450 pounds. A random sample of 16 adult hippos had the sample mean weight of 3,300 pounds and the sample standard deviation of 400 pounds. Assume that the two populations are approximately normally distributed. Construct a 95% confidence interval for the difference between their overall average weights of adult white rhinos and adult hippos. Assume that the overall standard deviations are equal.

$$s_{\text{pooled}}^2 = \frac{(9-1)\cdot 450^2 + (16-1)\cdot 400^2}{9+16-2} \approx 174{,}782.6 \qquad s_{\text{pooled}} \approx 418.07$$

$$\left(\overline{X} - \overline{Y}\right) \pm t_{\alpha/2} \cdot s_{\text{pooled}} \cdot \sqrt{\frac{1}{n} + \frac{1}{m}}$$

9 + 16 – 2 = **23** degrees of freedom

$$t_{0.025}(23) = 2.069 \qquad (5{,}100 - 3{,}300) \pm 2.069 \cdot 418.07 \cdot \sqrt{\frac{1}{9} + \frac{1}{16}}$$

1,800 ± 360.4 **(1,439.6, 2,160.4)**