

STAT 3202: Practice 02

Spring 2019, OSU

Exercise 1

Let X_1, X_2, \dots, X_n be iid $N(\theta, 1)$ and consider $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Show that \bar{X}_n is a consistent estimator of θ .

Solution:

First, note that $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ is an unbiased estimator as we have seen many times before.

$$E[\bar{X}_n] = E\left[\sum_{i=1}^n X_i\right] = \frac{1}{n} \cdot n\theta = \theta$$

Also, as we have seen before, the variance of \bar{X}_n is given by

$$\text{Var}[\bar{X}_n] = \frac{\sigma^2}{n} = \frac{1}{n}$$

Since we have an unbiased estimator, we simply need for the variance to vanish as n goes to infinity. We see that

$$\lim_{n \rightarrow \infty} \text{Var}[\bar{X}_n] = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Thus \bar{X}_n is a consistent estimator for θ .

Exercise 2

Suppose that X_1, X_2, \dots, X_n are an iid sample from the distribution

$$f(x; \theta) = \frac{1}{2}(1 + \theta x), \quad -1 < x < 1, -1 < \theta < 1.$$

Show that $3\bar{X}_n$ is a consistent estimator of θ .

Solution:

First, recall that we have previously calculated,

$$E[X_i] = \frac{\theta}{3}$$

and

$$\text{Var}[X_i] = \frac{3 - \theta^2}{9}$$

Thus, $3\bar{X}_n$ is an unbiased estimator of θ since

$$\text{E}[3\bar{X}_n] = 3 \cdot \text{E}[X_i] = 3 \cdot \frac{\theta}{3} = \theta$$

Then, we calculate the variance of the proposed estimator.

$$\text{Var}[3\bar{X}_n] = 9 \cdot \frac{\text{Var}[X_i]}{n} = 9 \cdot \frac{3 - \theta^2}{9n} = \frac{3 - \theta^2}{n}$$

Then, since $3\bar{X}_n$ is an unbiased estimator of θ and

$$\lim_{n \rightarrow \infty} \text{Var}[3\bar{X}_n] = \lim_{n \rightarrow \infty} \frac{3 - \theta^2}{n} = 0$$

we conclude that $3\bar{X}_n$ is a consistent estimator of θ .

Exercise 3

Let Y_1, Y_2, \dots, Y_n be a random sample such that

- $\text{E}[Y_i] = \mu$
- $\text{Var}[Y_i] = \sigma^2$.

Suggest a consistent estimator for μ^2 .

Solution:

First note that,

$$\text{E}[\bar{Y}_n] = \mu$$

and

$$\lim_{n \rightarrow \infty} \text{Var}[\bar{Y}_n] = \lim_{n \rightarrow \infty} \frac{\sigma^2}{n} = 0$$

Thus, \bar{Y}_n is a consistent estimator of μ .

Therefore, \bar{Y}_n^2 is a consistent estimator of μ^2 .

Exercise 4

Let X_1, X_2, \dots, X_n be iid $N(\mu_X, \sigma_X^2)$. Also, let Y_1, Y_2, \dots, Y_n be iid $N(\mu_Y, \sigma_Y^2)$.

Suggest a consistent estimator for $\mu_X - \mu_Y$.

Solution:

First note that \bar{X} is a consistent estimator for μ_X and \bar{Y} is a consistent estimator for μ_Y .

Because of this, we can say that $-\bar{Y}$ is a consistent estimator for $-\mu_Y$. Then finally we can conclude that $\bar{X} - \bar{Y}$ is a consistent estimator for $\mu_X - \mu_Y$.

Or, more directly note that

$$E[\bar{X} - \bar{Y}] = E[\bar{X}] - E[\bar{Y}] = \mu_X - \mu_Y$$

thus $\bar{X} - \bar{Y}$ is an unbiased estimator for $\mu_X - \mu_Y$.

Also note that,

$$\text{Var}[\bar{X} - \bar{Y}] = \frac{\text{Var}[\bar{X}]}{n} + \frac{\text{Var}[\bar{Y}]}{n} = \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{n}$$

Then, since

$$\lim_{n \rightarrow \infty} \text{Var}[\bar{X} - \bar{Y}] = \lim_{n \rightarrow \infty} \left(\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{n} \right) = 0$$

we can conclude that $\bar{X} - \bar{Y}$ is a consistent estimator for $\mu_X - \mu_Y$.

Exercise 5

Let X_1, X_2, \dots, X_n be iid $N(\mu_X, \sigma^2)$. Also, let Y_1, Y_2, \dots, Y_n be iid $N(\mu_Y, \sigma^2)$. Note that both distributions have the same variance.

Show that

$$\frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{2n - 2}$$

is a consistent estimator for σ^2 .

Hint: Note that

$$\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi_{n-1}^2$$

$$\frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2} \sim \chi_{n-1}^2$$

Also, recall that, if $W \sim \chi_k^2$, then $E[W] = k$ and $\text{Var}[W] = 2k$.

Solution:

First, using the hint, we can show that the estimator is unbiased.

$$\begin{aligned} \mathbb{E} \left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{2n - 2} \right] &= \mathbb{E} \left[\frac{\sigma^2 \cdot \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} + \sigma^2 \cdot \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2}}{2n - 2} \right] \\ &= \frac{\sigma^2}{2n - 2} \cdot \mathbb{E} \left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \right] + \frac{\sigma^2}{2n - 2} \cdot \mathbb{E} \left[\frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2} \right] \\ &= \frac{\sigma^2}{2n - 2} \cdot (n - 1) + \frac{\sigma^2}{2n - 2} \cdot (n - 1) \\ &= \frac{n\sigma^2 - \sigma^2 + n\sigma^2 - \sigma^2}{2n - 2} = \frac{2n - 2}{2n - 2} \cdot \sigma^2 = \sigma^2 \end{aligned}$$

Next, we calculate the variance.

$$\begin{aligned} \text{Var} \left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{2n - 2} \right] &= \text{Var} \left[\frac{\sigma^2 \cdot \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} + \sigma^2 \cdot \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2}}{2n - 2} \right] \\ &= \left(\frac{\sigma^2}{2n - 2} \right)^2 \cdot \text{Var} \left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \right] + \left(\frac{\sigma^2}{2n - 2} \right)^2 \cdot \text{Var} \left[\frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\sigma^2} \right] \\ &= \left(\frac{\sigma^2}{2n - 2} \right)^2 \cdot 2(n - 1) + \left(\frac{\sigma^2}{2n - 2} \right)^2 \cdot 2(n - 1) \\ &= \frac{2(\sigma^2)^2}{2n - 2} \end{aligned}$$

Then finally we see that

$$\lim_{n \rightarrow \infty} \text{Var} \left[\frac{\sum_{i=1}^n (X_i - \bar{X})^2 + \sum_{i=1}^n (Y_i - \bar{Y})^2}{2n - 2} \right] = \lim_{n \rightarrow \infty} \frac{2(\sigma^2)^2}{2n - 2} = 0$$

Thus the proposed estimator is consistent.

Exercise 6

Let Y_1, Y_2, \dots, Y_n be iid observations from a Poisson distribution with parameter λ . Show that $U = \sum_{i=1}^n Y_i$ is sufficient for λ .

Solution:

First, recall that the pmf for a Poisson random variable is given by

$$f(y | \lambda) = \frac{\lambda^y e^{-\lambda}}{y!}, \quad y = 0, 1, 2, \dots, \lambda > 0$$

Then we have

$$\begin{aligned} f(y_1, y_2, \dots, y_n | \lambda) &= \prod_{i=1}^n \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \\ &= \frac{\lambda^{\sum_{i=1}^n y_i} e^{-n\lambda}}{\prod_{i=1}^n y_i!} \\ &= \lambda^u e^{-n\lambda} \frac{1}{\prod_{i=1}^n y_i!} \end{aligned}$$

Here we have

$$\begin{aligned} U &= \sum_{i=1}^n Y_i \\ g(u, \lambda) &= \lambda^u e^{-n\lambda} \\ h(y_1, y_2, \dots, y_n) &= \frac{1}{\prod_{i=1}^n y_i!} \end{aligned}$$

Thus, by the Factorization Theorem,

$$U = \sum_{i=1}^n Y_i$$

is a sufficient statistic for λ .

Exercise 7

Let X_1, X_2, \dots, X_n be iid observations from a distribution with density

$$f(x | \theta) = \frac{\theta}{(1+x)^{\theta+1}}, \quad 0 < \theta < \infty, 0 < x < \infty$$

Find a sufficient statistic for θ .

Solution:

First note that

$$\begin{aligned}
f(x_1, x_2, \dots, x_n | \theta) &= \prod_{i=1}^n \frac{\theta}{(1+x_i)^{\theta+1}} \\
&= \frac{\theta^n}{\left[\prod_{i=1}^n (1+x_i) \right]^{\theta+1}} \\
&= \frac{\theta^n}{u^{\theta+1}}
\end{aligned}$$

Here we have

$$\begin{aligned}
U &= \prod_{i=1}^n (1+X_i) \\
g(u, \theta) &= \frac{\theta^n}{u^{\theta+1}} \\
h(x_1, x_2, \dots, x_n) &= 1
\end{aligned}$$

Thus, by the Factorization Theorem,

$$U = \prod_{i=1}^n (1+X_i)$$

is a sufficient statistic for θ .

Exercise 8

Let X_1, X_2, \dots, X_n be iid observations from a normal distribution with a unknown mean, μ , and known variance $\sigma^2 = 9$.

Show that $\sum_{i=1}^n X_i$ is a sufficient statistic for μ , then use this statistic to create an estimator that is both unbiased and sufficient for estimating μ .

Solution:

First recall that the pdf for a normal distribution is given by

$$f(x | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \right], \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma^2 > 0$$

In this case, with $\sigma^2 = 9$, we have

$$f(x | \mu) = \frac{1}{3\sqrt{2\pi}} \exp \left[-\frac{1}{18} (x - \mu)^2 \right]$$

Now note that

$$\begin{aligned}
f(x_1, x_2, \dots, x_n | \theta) &= \prod_{i=1}^n \frac{1}{3\sqrt{2\pi}} \exp \left[-\frac{1}{18} (x_i - \mu)^2 \right] \\
&= \left(\frac{1}{3\sqrt{2\pi}} \right)^n \prod_{i=1}^n \exp \left[-\frac{1}{18} (x_i^2 - 2\mu x_i + \mu^2) \right] \\
&= \left(\frac{1}{3\sqrt{2\pi}} \right)^n \exp \left[-\frac{1}{18} \sum_{i=1}^n x_i^2 \right] \exp \left[\frac{2\mu}{18} \sum_{i=1}^n x_i \right] \exp \left[-\frac{1}{18} n\mu^2 \right] \\
&= \left(\frac{1}{3\sqrt{2\pi}} \right)^n \exp \left[-\frac{1}{18} \sum_{i=1}^n x_i^2 \right] \exp \left[\frac{2\mu}{18} u \right] \exp \left[-\frac{1}{18} n\mu^2 \right]
\end{aligned}$$

Here we have

$$\begin{aligned}
U &= \sum_{i=1}^n X_i \\
g(u, \mu) &= \exp \left[\frac{2\mu}{18} u \right] \exp \left[-\frac{1}{18} n\mu^2 \right] \\
h(x_1, x_2, \dots, x_n) &= \left(\frac{1}{3\sqrt{2\pi}} \right)^n \exp \left[-\frac{1}{18} \sum_{i=1}^n x_i^2 \right]
\end{aligned}$$

Thus, by the Factorization Theorem,

$$U = \sum_{i=1}^n X_i$$

is a sufficient statistic for μ .

Then, because $f(u) = \frac{1}{n}u$ is a one-to-one function,

$$U^* = \frac{1}{n} \sum_{i=1}^n X_i$$

is also a sufficient statistic for μ , which also happens to be unbiased, which we have seen many times before.

Exercise 9

Let Y_1, Y_2, \dots, Y_n be iid observations from a distribution with density

$$f(y | \beta) = \frac{y}{\beta} \cdot \exp \left(\frac{-y^2}{2\beta} \right), \quad y \geq 0, \beta > 0$$

Find a sufficient statistic for β .

Solution:

First note that

$$\begin{aligned} f(y_1, y_2, \dots, y_n | \theta) &= \prod_{i=1}^n \frac{y_i}{\beta} \cdot \exp\left(\frac{-y_i^2}{2\beta}\right) \\ &= \beta^{-n} \left(\prod_{i=1}^n y_i\right) \exp\left[-\frac{\sum_{i=1}^n y_i^2}{2\beta}\right] \\ &= \beta^{-n} \left(\prod_{i=1}^n y_i\right) \exp\left[-\frac{u}{2\beta}\right] \end{aligned}$$

Here we have

$$U = \sum_{i=1}^n Y_i^2$$

$$g(u, \beta) = \beta^{-n} \exp\left[-\frac{u}{2\beta}\right]$$

$$h(y_1, y_2, \dots, y_n) = \prod_{i=1}^n y_i$$

Thus, by the Factorization Theorem,

$$U = \sum_{i=1}^n Y_i^2$$

is a sufficient statistic for β .

Exercise 10

Let X_1, X_2, \dots, X_n be iid observations from a distribution with density

$$f(x | \alpha, \beta) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha}, \quad x \geq 0, \alpha > 0, \beta > 0$$

Let α be a known constant and β be unknown. Find a sufficient statistic for β .

Solution:

First note that

$$\begin{aligned}
f(x_1, x_2, \dots, x_n | \theta) &= \prod_{i=1}^n \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha} \\
&= \alpha^n \beta^{-n} \frac{\prod_{i=1}^n x_i^{\alpha-1}}{\prod_{i=1}^n \beta^{\alpha-1}} \exp\left[-\frac{\sum_{i=1}^n x_i^\alpha}{\beta^\alpha}\right] \\
&= \alpha^n \beta^{-n} \left(\prod_{i=1}^n x_i\right)^{\alpha-1} \beta^{n-n\alpha} \exp\left[-\frac{\sum_{i=1}^n x_i^\alpha}{\beta^\alpha}\right] \\
&= \alpha^n \beta^{-n\alpha} \left(\prod_{i=1}^n x_i\right)^{\alpha-1} \exp\left[-\frac{\sum_{i=1}^n x_i^\alpha}{\beta^\alpha}\right] \\
&= \alpha^n \beta^{-n\alpha} \left(\prod_{i=1}^n x_i\right)^{\alpha-1} \exp\left[-\frac{u}{\beta^\alpha}\right]
\end{aligned}$$

Here we have

$$U = \sum_{i=1}^n X_i^\alpha$$

$$g(u, \beta) = \beta^{-n\alpha} \exp\left[-\frac{u}{\beta^\alpha}\right]$$

$$h(x_1, x_2, \dots, x_n) = \alpha^n \left(\prod_{i=1}^n x_i\right)^{\alpha-1}$$

Thus, by the Factorization Theorem,

$$U = \sum_{i=1}^n X_i^\alpha$$

is a sufficient statistic for β .
