STAT 3202: Practice 10

Spring 2019, OSU

Exercise 1

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 5, \beta = 5)$
- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim$?

and observed data with statistics,

- Sample size: $n = 20$
- Number of "successes" $\sum x_i = 15$

Use the given prior and the observed data to calculate a Bayes' estimate of θ. (Use the posterior mean.)

Solution

- Recall that a Beta distribution with parameters α and β has mean $\frac{\alpha}{\alpha+\beta}$.
- The posterior distribution for the Beta-Bernoulli model is a Beta distribution with parameters $a = \alpha + \sum x_i$ and $b = \beta + \sum y_i$.
  - Here we have, $\sum y_i = n - \sum x_i$.
  - Here we are using α and β as the parameters for the prior, and a and b as the parameters of the posterior.

So, in this case, we have

$$\theta \mid X_1, X_1, \ldots, X_n \sim \text{Beta}(a = 20, b = 10)$$

Then we have

$$\hat{\theta}_B = \frac{a}{a+b} = \frac{20}{20+10} = \boxed{\frac{2}{3}}$$

Exercise 2

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 50, \beta = 20)$
- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim$?

and observed data with statistics,

- Sample size: $n = 40$
- Number of "successes" $\sum x_i = 32$

Use the given prior and the observed data to calculate a Bayes' estimate of θ. (Use the posterior mean.)

Solution

In this case, we have

$$\theta \mid X_1, X_1, \ldots, X_n \sim \text{Beta}(a = 82, b = 28)$$

Then we have

$$\hat{\theta}_B = \frac{a}{a+b} = \frac{82}{82+28} = \boxed{\frac{41}{55}}$$

Exercise 3

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 10, \beta = 60)$
- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim$?

and observed data with statistics,

- Sample size: $n = 5$
- Number of "successes" $\sum x_i = 2$

Use the given prior and the observed data to calculate a Bayes' estimate of θ. (Use the posterior mean.)

Solution

In this case, we have

$$\theta \mid X_1, X_1, \ldots, X_n \sim \text{Beta}(a = 12, b = 63)$$

Then we have

$$\hat{\theta}_B = \frac{a}{a+b} = \frac{12}{12+63} = \boxed{\frac{4}{25}}$$

Exercise 4

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 10, \beta = 60)$
- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim$?

and observed data with statistics,

- Sample size: $n = 5$
- Number of "successes" $\sum x_i = 2$

Use the given prior and the observed data to calculate a 99% credible interval

Solution

```r
qbeta(c(0.005, 0.995), shape1 = 10 + 2, shape2 = 60 + 3)
```

```
## [1] 0.06973818 0.28374039
```

Exercise 5

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 4, \beta = 4)$
- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim ?$

and observed data with statistics,

- Sample size: $n = 5$
- Number of "successes" $\sum x_i = 2$

Use the given prior and the observed data to calculate a 90% credible interval

Solution

```r
qbeta(c(0.05, 0.95), shape1 = 4 + 2, shape2 = 4 + 3)
```

```
## [1] 0.2452998 0.6847622
```

Exercise 6

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 10, \beta = 4)$
- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim ?$

and observed data with statistics,

- Sample size: $n = 50$
- Number of "successes" $\sum x_i = 2$

Use the given prior and the observed data to calculate a 95% credible interval

Solution

```r
qbeta(c(0.025, 0.975), shape1 = 10 + 2, shape2 = 4 + 48)
```

```
## [1] 0.1024842 0.2909709
```

Exercise 7

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 10, \beta = 4)$

- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim ?$

and observed data with statistics,

- Sample size: $n = 50$
- Number of "successes" $\sum x_i = 20$

Use the given prior and the observed data to test $H_0 : \theta > 0.50$ vs $H_1 : \theta \leq 0.50$

Solution

```
pbeta(0.50, shape1 = 10 + 20, shape2 = 4 + 30, lower.tail = FALSE)
```

```
## [1] 0.3073275
```

```
pbeta(0.50, shape1 = 10 + 20, shape2 = 4 + 30)
```

```
## [1] 0.6926725
```

Accept H_1.

Exercise 8

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 10, \beta = 10)$
- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim ?$

and observed data with statistics,

- Sample size: $n = 20$
- Number of "successes" $\sum x_i = 5$

Use the given prior and the observed data to test $H_0 : 0.25 < \theta < 0.50$ vs $H_1 : \theta \leq 0.25, \theta \geq 0.50$

Solution

```
diff(pbeta(c(0.25, 0.50), shape1 = 10 + 5, shape2 = 10 + 15))
```

```
## [1] 0.9020704
```

Accept H_0.

Exercise 9

Consider the following model,

- Prior: $\theta \sim \text{Beta}(\alpha = 3, \beta = 3)$
- Likelihood: $X_1, X_1, \ldots, X_n \sim \text{Bern}(\theta)$
- Posterior: $\theta \mid X_1, X_1, \ldots, X_n \sim ?$

and observed data with statistics,

- Sample size: $n = 100$
- Number of "successes" $\sum x_i = 70$

Use the given prior and the observed data to test $H_0 : \theta > 0.80$ vs $H_1 : \theta \leq 0.80$

Solution

```
pbeta(0.80, shape1 = 3 + 70, shape2 = 3 + 30, lower.tail = FALSE)
```

```
## [1] 0.00374593
```

```
pbeta(0.80, shape1 = 3 + 70, shape2 = 3 + 30)
```

```
## [1] 0.9962541
```

Accept H_1.
