

Estimation II: Consistency

Stat 3202 @ OSU

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Suppose we are interested in the value of some parameter θ that describes a feature of a population. We draw a random sample from the population, X_1, \dots, X_n , and have an estimator $\hat{\theta}$ which is a function of the sample: $\hat{\theta} = \hat{\theta}(X_1, \dots, X_n)$.

- *Idea:* We'd like $\hat{\theta}$ to get “closer” and closer to θ as we draw larger and larger samples.

Definition: Statistical Consistency

An estimator $\hat{\theta}_n$ is said to be a **consistent estimator** of θ if, for any positive ϵ ,

$$\lim_{n \rightarrow \infty} P(|\hat{\theta}_n - \theta| \leq \epsilon) = 1$$

or, equivalently,

$$\lim_{n \rightarrow \infty} P(|\hat{\theta}_n - \theta| > \epsilon) = 0$$

We say that $\hat{\theta}_n$ **converges in probability** to θ and we write $\hat{\theta}_n \xrightarrow{P} \theta$.

Example: Using the Definition

An estimator $\hat{\theta}_n$ is said to be a **consistent estimator** of θ if, for any positive ϵ ,

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- Let X_1, X_2, \dots, X_n be iid $N(\theta, 1)$ and consider $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Use the definition of consistency to show that \bar{X}_n is a consistent estimator of θ .

Theorem: An *unbiased* estimator $\hat{\theta}_n$ for θ is a consistent estimator of θ if

$$\lim_{n \rightarrow \infty} \text{Var} [\hat{\theta}_n] = 0$$

- Proof?

Theorem: An *unbiased* estimator $\hat{\theta}_n$ for θ is a consistent estimator of θ if

$$\lim_{n \rightarrow \infty} \text{Var} [\hat{\theta}_n] = 0$$

- Again letting X_1, X_2, \dots, X_n be iid $N(\theta, 1)$ and consider $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Show that \bar{X}_n is a consistent estimator of θ .

Example

Suppose that X_1, X_2, \dots, X_n are an iid sample from the distribution

$$f(x; \theta) = \frac{1}{2}(1 + \theta x), \quad -1 < x < 1, -1 < \theta < 1.$$

Previously:

- $\hat{\theta} = 3\bar{X}_n$ is an unbiased estimator of θ .

Is $3\bar{X}_n$ a consistent estimator of θ ?

The (Weak) Law of Large Numbers

Let Y_1, Y_2, \dots, Y_n be a random sample such that

- $E[Y_i] = \mu$
- $\text{Var}[Y_i] = \sigma^2$.

Show that $\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i$ is a consistent estimator of μ . Thus, show that

$$\bar{Y}_n \xrightarrow{P} \mu$$

Theorem: Suppose that $\hat{\theta}_n \xrightarrow{P} \theta$ and that $\hat{\beta}_n \xrightarrow{P} \beta$.

- $\hat{\theta}_n + \hat{\beta}_n \xrightarrow{P} \theta + \beta$
- $\hat{\theta}_n \times \hat{\beta}_n \xrightarrow{P} \theta \times \beta$
- $\hat{\theta}_n \div \hat{\beta}_n \xrightarrow{P} \theta \div \beta$
- If $g(\cdot)$ is a real valued function that is continuous at θ , then $g(\hat{\theta}_n) \xrightarrow{P} g(\theta)$

Example

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- If $g(\cdot)$ is a real valued function that is continuous at θ , then $g(\hat{\theta}_n) \xrightarrow{P} g(\theta)$

Let Y_1, Y_2, \dots, Y_n be a random sample such that

- $E[Y_i] = \mu$
- $\text{Var}[Y_i] = \sigma^2$.

Suggest a consistent estimator for μ^2 .

Example

Theorem: Suppose that $\hat{\theta}_n \xrightarrow{P} \theta$ and that $\hat{\beta}_n \xrightarrow{P} \beta$.

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- $\hat{\theta}_n \times \hat{\beta}_n \xrightarrow{P} \theta \times \beta$
- $\hat{\theta}_n \div \hat{\beta}_n \xrightarrow{P} \theta \div \beta$
- If $g(\cdot)$ is a real valued function that is continuous at θ , then $g(\hat{\theta}_n) \xrightarrow{P} g(\theta)$

Let X_1, X_2, \dots, X_n be iid $N(\mu_X, \sigma_X^2)$. Also, let Y_1, Y_2, \dots, Y_m be iid $N(\mu_Y, \sigma_Y^2)$.

Suggest a consistent estimator for $\mu_X - \mu_Y$.