Estimation III: Method of Moments and Maximum Likelihood

Stat 3202 @ OSU

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A Standard Setup

Let $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

How should we estimate λ?

Population and Sample Moments

The k^{th} **population moment** of a RV (about the origin) is

$$\mu_k^{'} = \mathrm{E}\left[Y^k\right]$$

The k^{th} **sample moment** is

$$m_k^{'} = \overline{Y^k} = \frac{1}{n}\sum_{i=1}^{n} Y_i^k$$

The Method of Moments (MoM)

The **Method of Moments** (MoM) consists of equating sample moments and population moments. If a population has t parameters, the MOM consists of solving the system of equations

$$m_k^{'} = \mu_k^{'}, \;\; k = 1, 2, \ldots, t$$

for the t parameters.

Example: Poisson

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

Find a method of moments estimator of λ, call it $\tilde{\lambda}$.

Example: Normal, Two Unknowns

Let $X_1, X_2, \ldots, X_n$ be iid $N(\theta, \sigma^2)$.

Use the method of moments to estimate the *parameter vector* $\left(\theta, \sigma^2\right)$.

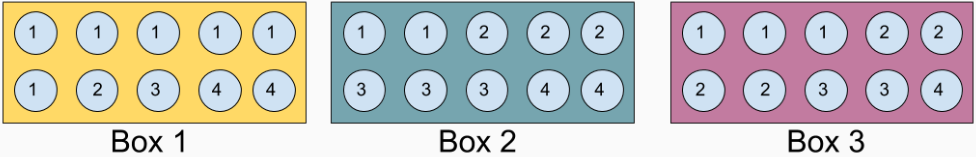
Example: Normal, Mean Known

Let $X_1, X_2, \ldots, X_n$ be iid $N(1, \sigma^2)$.

Find a method of moments estimator of σ^2, call it $\tilde{\sigma}^2$.

Calculus???

A Game Show / An Idea



Is a Coin Fair?

Let $Y \sim \text{binom}(n = 100, p)$.

Suppose we observe a single observation $x = 60$.

Log Rules

- $x^m x^n = x^{m+n}$
- $(x^m)^n = x^{mn}$
- $\log(ab) = \log(a) + \log(b)$
- $\log(a/b) = \log(a) - \log(b)$
- $\log(a^b) = b\log(a)$
- $\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot \cdots \cdot x_n$
- $\prod_{i=1}^{n} x_i^a = \left(\prod_{i=1}^{n} x_i\right)^a$
- $\log\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \log(x_i)$

Example: Poisson

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$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

Find the maximum likelihood estimator of λ, call it $\hat{\lambda}$.

Example: Poisson

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

Calculate the maximum likelihood estimate of λ, when

$$x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 2.$$

Maximum Likelihood Estimation (MLE)

Given a random sample $X_1, X_2, \ldots, X_n$ from a population with parameter θ and density or mass $f(x \mid \theta)$, we have:

The Likelihood, $L(\theta)$,

$$L(\theta) = f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i \mid \theta)$$

The **Maximum Likelihood Estimator**, $\hat{\theta}$

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\ L(\theta) = \underset{\theta}{\operatorname{argmax}}\ \log L(\theta)$$

Invariance Principle

If $\hat{\theta}$ is the MLE of θ and the function $h(\theta)$ is continuous, then $h(\hat{\theta})$ is the MLE of $h(\theta)$.

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. That is

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \quad \lambda > 0$$

- **Example:** Find the maximum likelihood estimator of $P[X = 4]$, call it $\hat{P}[X = 4]$. Calculate an **estimate** using this *estimator* when

$$x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 2.$$

Some Brief History

Who Is This?

Who Is This?

Another Example

Let $X_1, X_2, \ldots, X_n$ iid from a population with pdf

$$f(x \mid \theta) = \frac{1}{\theta} x^{(1-\theta)/\theta}, \quad 0 < x < 1,\ 0 < \theta < \infty$$

Find the maximum likelihood estimator of θ, call it $\hat{\theta}$.

A Different Example

Let $X_1, X_2, \ldots, X_n$ iid from a population with pdf

$$f(x \mid \theta) = \frac{\theta}{x^2}, \quad 0 < \theta \leq x < \infty$$

Find the maximum likelihood estimator of θ, call it $\hat{\theta}$.

Example: Gamma

Let $X_1, X_2, \ldots, X_n \sim$ iid gamma(α, β) with α known.

Find the maximum likelihood estimator of β, call it $\hat{\beta}$.

More Practice

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a distribution with pdf

$$f(y \mid \alpha) = \frac{2}{\alpha} \cdot y \cdot \exp\left\{-\frac{y^2}{\alpha}\right\}, \quad y > 0, \quad \alpha > 0.$$

Find the maximum likelihood **estimator** of α.

More Practice

Suppose that a random variable X follows a discrete distribution, which is determined by a parameter θ which can take *only two values*, $\theta = 1$ or $\theta = 2$. The parameter θ is unknown. If $\theta = 1$, then X follows a Poisson distribution with parameter $\lambda = 2$. If $\theta = 2$, then X follows a Geometric distribution with parameter $p = 0.25$. Now suppose we observe $X = 3$. Based on this data, what is the maximum likelihood **estimate** of θ?

More More More

Let $Y_1, Y_2, \ldots, Y_n \overset{\text{iid}}{\sim} f(y \mid \theta) = \begin{cases} \dfrac{2\theta^2}{y^3} & \theta \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$

Find the maximum likelihood **estimator** of θ.

Next Time

- More examples?
- Why does this work?
- Why do we need both MLE and MoM?
- How do we use these methods in practice?