STAT 400
UIUC

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Practice Problems #1

SOLUTIONS

The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1 – 4. Do NOT use a computer. You may only use +, –, ×, ÷, and $\sqrt{\ }$ on a calculator. Show all work.

1. Evaluate the following integrals:

a) $\displaystyle\int_0^\infty x^2 e^{-3x}\,dx$; b) $\displaystyle\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)dx$; c) $\displaystyle\int_1^\infty \frac{x}{(1+x)^4}\,dx$.

a) $$\int_0^\infty x^2 e^{-3x}\,dx = \left(-\frac{1}{3}\cdot x^2\cdot e^{-3x} - \frac{2}{3^2}\cdot x\cdot e^{-3x} - \frac{2}{3^3}\cdot e^{-3x}\right)\Bigg|_0^\infty = \mathbf{\frac{2}{27}}.$$

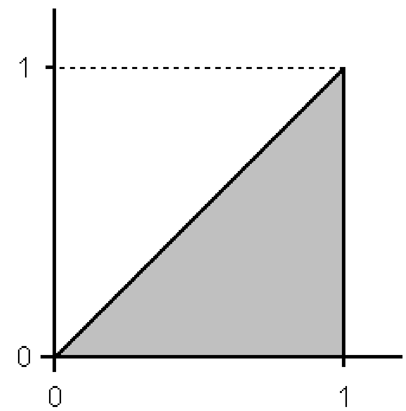
b) $$\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)dx = \int_0^{\sqrt{\pi}} \frac{1}{2}\sin\left(x^2\right)(2x\,dx) = \left(-\frac{\cos\left(x^2\right)}{2}\right)\Bigg|_0^{\sqrt{\pi}} = \mathbf{1}.$$

c) $$\int_1^\infty \frac{x}{(1+x)^4}\,dx = \int_2^\infty \frac{(u-1)}{u^4}\,du = \int_2^\infty \frac{1}{u^3}\,du - \int_2^\infty \frac{1}{u^4}\,du = \frac{1}{8} - \frac{1}{24} = \mathbf{\frac{1}{12}}.$$

2. Let $c > 0$. Consider $f(x, y) = x + c\,y$ and $\mathrm{A} = \{(x, y):\ 0 < y < x < 1\}$.

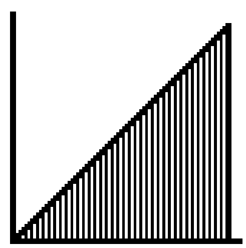
a) Sketch A. That is, sketch

$$\{(x, y):\ 0 < y < x < 1\}.$$



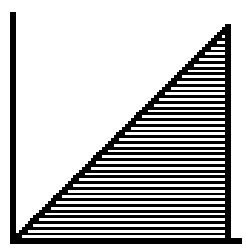
b) Set up the double integral(s) of $f(x, y)$ over A with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^1 \left(\int_0^x (x + c\,y)\,dy \right) dx$$



c) Set up the double integral(s) of $f(x, y)$ over A with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1 \left(\int_y^1 (x + c\,y)\,dx \right) dy$$



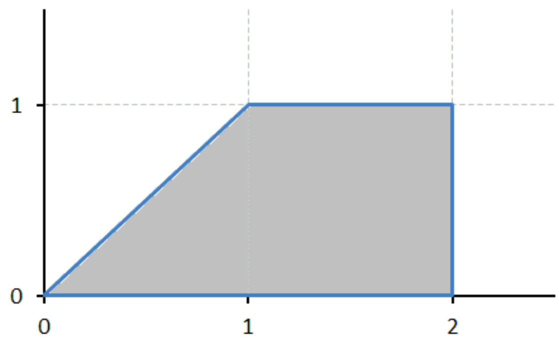
d) Find the value of c such that $\iint\limits_{\mathrm{A}} f(x, y)\,dx\,dy = 1$.

$$\int_0^1 \left(\int_0^x (x + c\,y)\,dy \right) dx = \int_0^1 \left(1 + \frac{c}{2}\right) x^2\,dx = \frac{1}{3}\left(1 + \frac{c}{2}\right) = \frac{1}{3} + \frac{c}{6} = 1.$$

$\Rightarrow \quad c = \mathbf{4}.$

3. Consider $f(x,y) = x\,y^3$ and $A = \{\,(x,y):\ 0 < y < 1,\ y < x < 2\,\}$.

a) Sketch A. That is, sketch $\{\,(x,y):\ 0 < y < 1,\ y < x < 2\,\}$.



b) Set up the double integral(s) of $f(x,y)$ over A with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^1 \left(\int_0^x x\,y^3\,dy \right) dx \;+\; \int_1^2 \left(\int_0^1 x\,y^3\,dy \right) dx$$

c) Set up the double integral(s) of $f(x,y)$ over A with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1 \left(\int_y^2 x\,y^3\,dx \right) dy$$

d) Use either (b) or (c) to evaluate $\displaystyle\iint_A f(x,y)\,dx\,dy$.

$$\int_0^1 \left(\int_y^2 x\,y^3\,dx \right) dy \;=\; \int_0^1 \left(\frac{1}{2}x^2\,y^3 \right)\Bigg|_{x=y}^{x=2} dy \;=\; \int_0^1 \left(2\,y^3 - \frac{1}{2}y^5 \right) dy$$

$$= \left(\frac{1}{2}y^4 - \frac{1}{12}y^6 \right)\Bigg|_{y=0}^{y=1} \;=\; \frac{1}{2} - \frac{1}{12} \;=\; \mathbf{\frac{5}{12}}\,.$$

OR

$$\int_0^1\left(\int_0^x x\,y^3\,dy\right)dx \;+\; \int_1^2\left(\int_0^1 x\,y^3\,dy\right)dx \;=\; \int_0^1 \frac{x^5}{4}dx \;+\; \int_1^2 \frac{x}{4}dx$$

$$= \frac{1}{24}+\left(\frac{1}{2}-\frac{1}{8}\right) \;=\; \mathbf{\frac{5}{12}}\,.$$

4. Evaluate the following sums $\left(\text{do NOT use a calculator}\right)$:

a) $\displaystyle\sum_{k=3}^{\infty} \frac{2}{5^{k}}$; b) $\displaystyle\sum_{k=1}^{\infty} 0.6^{2k+1}$; c) $\displaystyle\sum_{k=5}^{\infty} \frac{2^{k}}{k!}$.

a) $$\sum_{k=3}^{\infty} \frac{2}{5^{k}} = \frac{2}{5^{3}} + \frac{2}{5^{4}} + \frac{2}{5^{5}} + \frac{2}{5^{6}} + \ldots$$

Geometric series. Base = $\frac{1}{5}$.

$$\sum_{k=3}^{\infty} \frac{2}{5^{k}} = \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{2}{5^{3}}}{1 - \frac{1}{5}} = \frac{\frac{2}{125}}{\frac{4}{5}} = \mathbf{\frac{1}{50}} = \mathbf{0.02}.$$

b) $$\sum_{k=1}^{\infty} 0.6^{2k+1} = 0.6^{3} + 0.6^{5} + 0.6^{7} + 0.6^{9} + \ldots$$

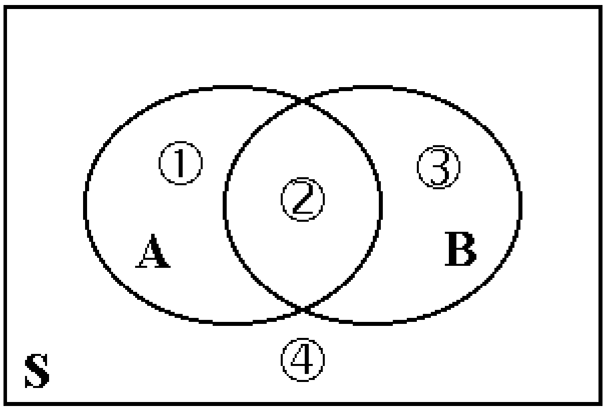
Geometric series. Base = $0.6^{2} = 0.36$.

$$\sum_{k=1}^{\infty} 0.6^{2k+1} = \frac{\textit{first term}}{1 - \textit{base}} = \frac{0.6^{3}}{1 - 0.36} = \frac{0.216}{0.64} = \mathbf{\frac{27}{80}} = \mathbf{0.3375}.$$

c) $$\sum_{k=5}^{\infty} \frac{2^{k}}{k!} = \sum_{k=0}^{\infty} \frac{2^{k}}{k!} - \frac{2^{0}}{0!} - \frac{2^{1}}{1!} - \frac{2^{2}}{2!} - \frac{2^{3}}{3!} - \frac{2^{4}}{4!} = e^{2} - 1 - 2 - 2 - \frac{4}{3} - \frac{2}{3} = \boldsymbol{e^{2} - 7}.$$

5. Suppose $P(A) = 0.70$, $P(B) = 0.50$, $P(A \cup B') = 0.80$.

a) Find $P(A \cap B)$. b) Find $P(A \cup B)$. c) Find $P(A \mid B)$.



$P(A)$ = ① + ② = 0.70,

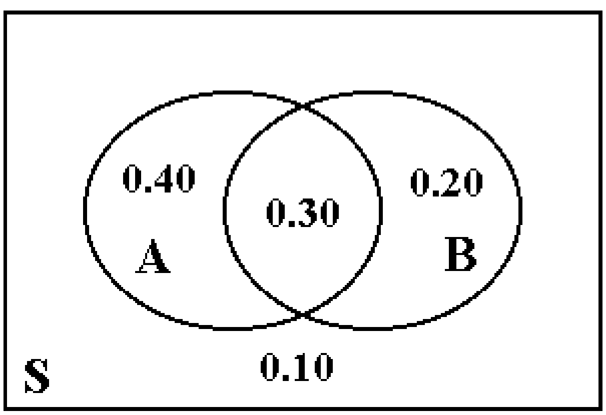
$P(B)$ = ② + ③ = 0.50,

$P(A \cup B')$ = ① + ② + ④ = 0.80.

$\Rightarrow$ ④ = 0.10.

$P(B')$ = ① + ④ = $1 - P(B)$ = 0.50, $\Rightarrow$ ① = 0.40.

$\Rightarrow$ ② = 0.30. $\Rightarrow$ ③ = 0.20.



a) $P(A \cap B) = \mathbf{0.30}$.

b) $P(A \cup B) = \mathbf{0.90}$.

c) $P(A \mid B) = \dfrac{0.30}{0.50} = \mathbf{0.60}$.

6. Suppose

$P(A) = 0.40$,

$P(B) = 0.34$,

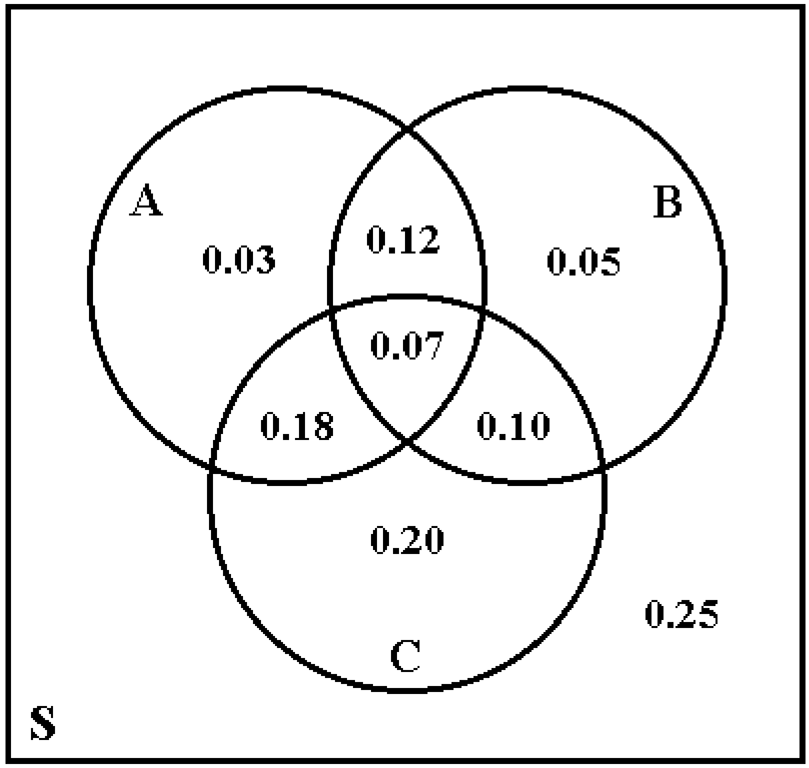
$P(C) = 0.55$,

$P(A \cap B) = 0.19$,

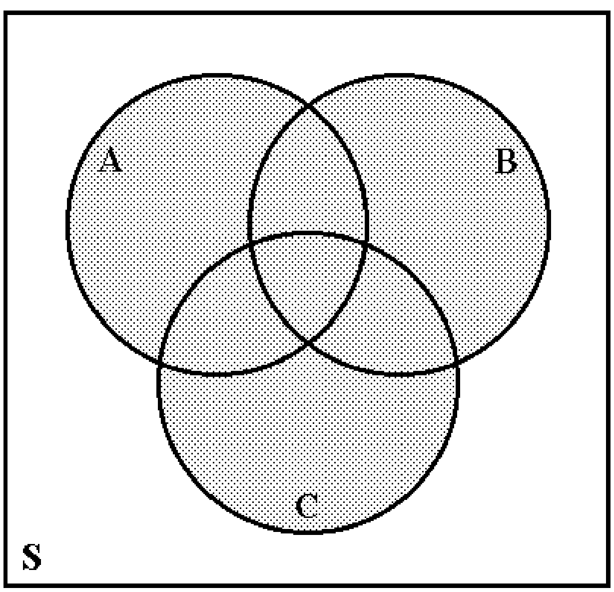
$P(A \cap C) = 0.25$,

$P(B \cap C) = 0.17$,

$P(A \cap B \cap C) = 0.07$.

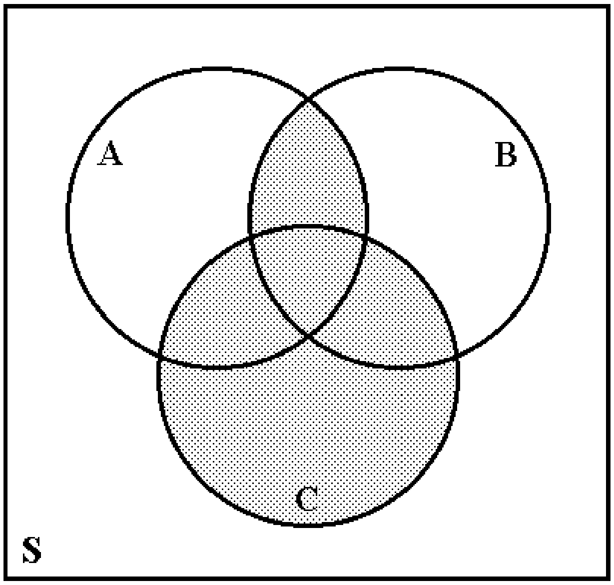


a) Find $P(A \cup B \cup C)$.



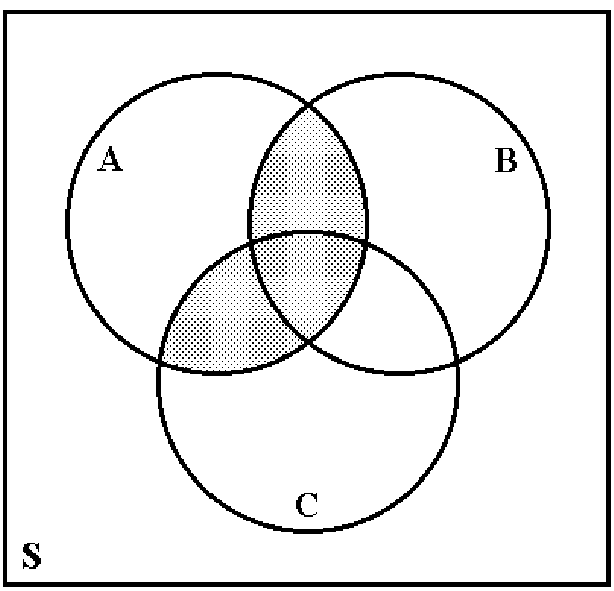
$P(A \cup B \cup C) =$ **0.75**.

b) Find $P((A \cap B) \cup C)$.



$P((A \cap B) \cup C)$ = **0.67**.

c) Find $P(A \cap (B \cup C))$.



$P(A \cap (B \cup C))$ = **0.37**.

7. Let $a > 1$. Suppose $S = \{ 2, 3, 4, 5, 6, \dots \}$ and $P(k) = \dfrac{c}{a^k}$, $k = 2, 3, 4, 5, 6, \dots$.

a) Find the value of c that makes this is a valid probability distribution.

Must have $\sum\limits_{\text{all } x} P(x) = 1.$

$$\sum_{k=2}^{\infty} \frac{c}{a^k} = \frac{c}{a^2} + \frac{c}{a^3} + \frac{c}{a^4} + \frac{c}{a^5} + \dots = \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{c}{a^2}}{1 - \frac{1}{a}} = \frac{c}{a(a-1)} = 1.$$

$$\Rightarrow \qquad c = a(a-1).$$

b) Find $P(\text{outcome is odd})$.

$$P(\text{outcome is odd}) = P(3) + P(5) + P(7) + P(9) + \dots$$

$$= \frac{a(a-1)}{a^3} + \frac{a(a-1)}{a^5} + \frac{a(a-1)}{a^7} + \frac{a(a-1)}{a^9} + \dots$$

$$= \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{a-1}{a^2}}{1 - \frac{1}{a^2}} = \frac{a-1}{a^2 - 1} = \frac{1}{a+1}.$$

OR

$$P(\text{outcome is odd}) = P(3) + P(5) + P(7) + P(9) + \dots$$

$$= \frac{a(a-1)}{a^3} + \frac{a(a-1)}{a^5} + \frac{a(a-1)}{a^7} + \frac{a(a-1)}{a^9} + \dots$$

$$\text{P}(\text{outcome is even}) = \text{P}(2) + \text{P}(4) + \text{P}(6) + \text{P}(8) + \ldots$$

$$= \frac{a(a-1)}{a^2} + \frac{a(a-1)}{a^4} + \frac{a(a-1)}{a^6} + \frac{a(a-1)}{a^8} + \ldots$$

$$= a\,\text{P}(\text{outcome is odd}).$$

$$\text{P}(\text{outcome is odd}) + \text{P}(\text{outcome is even}) = 1.$$

$$\Rightarrow \quad \text{P}(\text{outcome is odd}) + a\,\text{P}(\text{outcome is odd}) = 1.$$

$$\Rightarrow \quad \text{P}(\text{outcome is odd}) = \frac{1}{a+1}.$$

c) Find $\text{P}(\text{outcome is less than or equal to } 5)$.

$$\text{P}(\text{outcome is less than or equal to } 5) = \text{P}(2) + \text{P}(3) + \text{P}(4) + \text{P}(5)$$

$$= \frac{a(a-1)}{a^2} + \frac{a(a-1)}{a^3} + \frac{a(a-1)}{a^4} + \frac{a(a-1)}{a^5}$$

$$= \frac{(a-1)}{a^4}\left(a^3 + a^2 + a + 1\right) = \frac{a^4 - 1}{a^4} = 1 - \frac{1}{a^4}.$$

OR

$$\text{P}(\text{outcome is less than or equal to } 5) = 1 - [\,\text{P}(6) + \text{P}(7) + \text{P}(8) + \text{P}(9) + \ldots\,]$$

$$= 1 - \left[\frac{a(a-1)}{a^6} + \frac{a(a-1)}{a^7} + \frac{a(a-1)}{a^8} + \frac{a(a-1)}{a^9} + \ldots\right]$$

$$= 1 - \frac{\textit{first term}}{1 - \textit{base}} = 1 - \frac{\frac{a-1}{a^5}}{1 - \frac{1}{a}} = 1 - \frac{1}{a^4}.$$

8. Suppose $S = \{ 2, 3, 4, 5, 6, \ldots \}$ and $P(k) = c \, \dfrac{2^k}{k!}$, $k = 2, 3, 4, 5, 6, \ldots$.

a) Find the value of c that makes this is a valid probability distribution.

Must have $\sum\limits_{\text{all } x} P(x) = 1$. $\Rightarrow$ $\sum\limits_{k=2}^{\infty} c \dfrac{2^k}{k!} = c \sum\limits_{k=2}^{\infty} \dfrac{2^k}{k!} = 1.$

$$\sum_{k=2}^{\infty} \frac{2^k}{k!} = \sum_{k=0}^{\infty} \frac{2^k}{k!} - 1 - 2 = e^2 - 3.$$

$$c = \frac{1}{e^2 - 3} \approx 0.22784.$$

b) Find $P(\text{outcome is greater than or equal to } 5)$.

$$1 - P(2) - P(3) - P(4) = 1 - \frac{1}{e^2 - 3} \left[\frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} \right]$$

$$= 1 - \frac{1}{e^2 - 3} \left[2 + \frac{4}{3} + \frac{2}{3} \right] = 1 - \frac{4}{e^2 - 3} \approx 0.08864.$$

OR

$$P(5) + P(6) + P(7) + P(8) + \ldots = \sum_{k=5}^{\infty} c \frac{2^k}{k!} = c \sum_{k=5}^{\infty} \frac{2^k}{k!}$$

$$= c \left[\sum_{k=0}^{\infty} \frac{2^k}{k!} - \frac{2^0}{0!} - \frac{2^1}{1!} - \frac{2^2}{2!} - \frac{2^3}{3!} - \frac{2^4}{4!} \right]$$

$$= c \left[e^2 - 1 - 2 - 2 - \frac{4}{3} - \frac{2}{3} \right] = = c \left\lfloor e^2 - 7 \right\rfloor = \frac{e^2 - 7}{e^2 - 3} \approx 0.08864.$$

9. A family that owns two automobiles is selected at random. Suppose that the probability that the older car is American is 0.80, the probability that the newer car is American is 0.70, and the probability that both the older and the newer cars are American is 0.60.

	New American	New Foreign	
Old American	**0.60**	0.20	**0.80**
Old Foreign	0.10	0.10	0.20
	0.70	0.30	1.00

a) Find the probability that at least one car is American (i.e. that either the older car or the newer car, or both cars are American).

$$P(\,OA \cup NA\,) = P(\,OA\,) + P(\,NA\,) - P(\,OA \cap NA\,) = 0.80 + 0.70 - 0.60 = \mathbf{0.90}.$$

b) What is the probability that the newer car is American, given that the older car is American?

$$P(\,NA \mid OA\,) = \frac{P(\,OA \cap NA\,)}{P(\,OA\,)} = \frac{0.60}{0.80} = \mathbf{0.75}.$$

c) What is the probability that the newer car is American, given that the older car is not American?

$$P(\,NA \mid OF\,) = \frac{P(\,OF \cap NA\,)}{P(\,OF\,)} = \frac{0.10}{0.20} = \mathbf{0.50}.$$

10. At Anytownville College, 25% of all students are enrolled in an introductory level statistics class, and 20% of all students are seniors. It is also known that 15% of the seniors are enrolled in an introductory level statistics class.

P(STAT) = 0.25, P(SR) = 0.20, P(STAT | SR) = 0.15.

a) *Initech* is offering an internship to a Anytownville College student who is either a senior or is enrolled in an introductory level statistics class, or both. What proportion of Anytownville College students are eligible for this internship?

Need $P(\text{SR} \cup \text{STAT}) = P(\text{SR}) + P(\text{STAT}) - P(\text{SR} \cap \text{STAT})$.

$P(\text{SR} \cap \text{STAT}) = P(\text{SR}) \times P(\text{STAT} \mid \text{SR}) = 0.20 \times 0.15 = 0.03$.

(3% of all students are seniors and are enrolled in an introductory level statistics class)

$P(\text{SR} \cup \text{STAT}) = P(\text{SR}) + P(\text{STAT}) - P(\text{SR} \cap \text{STAT}) = 0.20 + 0.25 - 0.03 = \mathbf{0.42}$.

	STAT	STAT′	
SR	**0.03**	0.17	**0.20**
SR′	0.22	0.58	0.80
	0.25	0.75	1.00

b) What proportion of students enrolled in an introductory level statistics class at Anytownville College are seniors?

$$P(\text{SR} \mid \text{STAT}) = \frac{P(\text{SR} \cap \text{STAT})}{P(\text{STAT})} = \frac{0.03}{0.25} = \mathbf{0.12}.$$

c) Given that a student selected at random is not a senior, what is the probability that he/she is enrolled in an introductory level statistics class?

$$P(\text{STAT} \mid \text{SR}') = \frac{P(\text{SR}' \cap \text{STAT})}{P(\text{SR}')} = \frac{0.22}{0.80} = \mathbf{0.275}.$$