Practice Problems #2

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SOLUTIONS

The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1. P(happy | dinner) = 0.90,

P(not happy | no dinner) = 0.60,

P(dinner) = 0.70.

Find …

a) P(dinner | happy);

b) P(no dinner | not happy).

a) $$P(\text{dinner} \mid \text{happy}) = \frac{0.70 \times 0.90}{0.70 \times 0.90 + 0.30 \times 0.40} = \frac{0.63}{0.75} = \mathbf{\frac{21}{25}} = \mathbf{0.84}.$$

b) $$P(\text{no dinner} \mid \text{not happy}) = \frac{0.30 \times 0.60}{0.70 \times 0.10 + 0.30 \times 0.60} = \frac{0.18}{0.25} = \mathbf{\frac{18}{25}} = \mathbf{0.72}.$$

	happy	not happy	Total
dinner	0.63	0.07	0.70
no dinner	0.12	0.18	0.30
Total	0.75	0.25	1.00

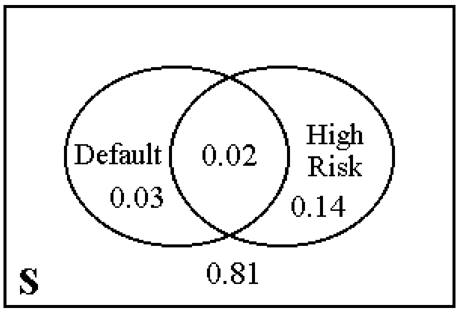
2. A bank classifies borrowers as "high risk" or "low risk," and 16% of its loans are made to those in the "high risk" category. Of all the bank's loans, 5% are in default. It is also known that 40% <u>of the loans in default</u> are to high-risk borrowers.

P(High risk) = 0.16, P(Default) = 0.05, P(High risk | Default) = 0.40.

a) What is the probability that a randomly selected loan is in default <u>and</u> issued to a high-risk borrower?

$$P(\text{Default} \cap \text{High risk}) = P(\text{Default}) \cdot P(\text{High risk} \mid \text{Default}) = 0.05 \cdot 0.40 = \mathbf{0.02}.$$

	High Risk	Low Risk	
Default	**0.02**	0.03	**0.05**
Default′	0.14	0.81	0.95
	0.16	0.84	1.00



b) What is the probability that a loan will default, <u>given that it is issued to a high-risk borrower</u>?

$$P(\text{Default} \mid \text{High risk}) = \frac{P(\text{Default} \cap \text{High risk})}{P(\text{High risk})} = \frac{0.02}{0.16} = \mathbf{0.125}.$$

c) What is the probability that a randomly selected loan is either in default <u>or</u> issued to a high-risk borrower, or both?

$$P(\text{Default} \cup \text{High risk}) = P(\text{Default}) + P(\text{High risk}) - P(\text{Default} \cap \text{High risk})$$

$$= 0.05 + 0.16 - 0.02 = \mathbf{0.19}.$$

d) A loan is being issued to a borrower who is not high-risk. What is the probability that this loan will default?

$$\text{P(Default} \mid \text{High risk}') = \frac{\text{P(Default} \cap \text{High risk}')}{\text{P(High risk}')} = \frac{0.03}{0.84} \approx \mathbf{0.0357}.$$

e) Are events {a randomly selected loan is in default} and {a randomly selected loan is issued to a high-risk borrower} independent? ***Justify your answer***.

P(High risk | Default) = 0.40. P(High risk) = 0.16.

Since P(High risk | Default) ≠ P(High risk), {Default} and {High risk} are **NOT independent**.

OR

P(Default | High risk) = 0.125. P(Default) = 0.05.

Since P(Default | High risk) ≠ P(Default), {Default} and {High risk} are **NOT independent**.

OR

P(Default ∩ High risk) = 0.02.

P(Default) × P(High risk) = 0.05 × 0.16 = 0.008.

Since P(Default ∩ High risk) ≠ P(Default) × P(High risk), {Default} and {High risk} are **NOT independent**.

3 – 4. An automobile insurance company classifies each driver as a good risk, a medium risk, or a poor risk. Of those currently insured, 30% are good risks, 50% are medium risks, and 20% are poor risks. In any given year, the probability that a driver will have a traffic accident is 0.1 for a good risk, 0.3 for a medium risk, and 0.5 for a poor risk.

P(Good) = 0.30, P(Accident | Good) = 0.10.

P(Medium) = 0.50, P(Accident | Medium) = 0.30.

P(Poor) = 0.20, P(Accident | Poor) = 0.50.

3. a) What is the probability that a randomly selected driver insured by this company had a traffic accident during 2014?

	Accident	No Accident		
Good	0.30 · 0.10 0.03	0.27	0.30	P(Accident | Good) = 0.10.
Medium	0.50 · 0.30 0.15	0.35	0.50	P(Accident | Medium) = 0.30.
Poor	0.20 · 0.50 0.10	0.10	0.20	P(Accident | Poor) = 0.50.
	0.28	0.72	1.00	

b) If a randomly selected driver insured by this company had a traffic accident during 2014, what is the probability that the driver is actually a poor risk?

$$P(\text{Poor} \mid \text{Accident}) = \frac{0.10}{0.28} = \mathbf{\frac{5}{14}} \approx \mathbf{0.357143}.$$

c) If a randomly selected driver insured by this company did not have a traffic accident during 2014, what is the probability that the driver is actually a good risk?

$$P(\text{Good} \mid \text{No Accident}) = \frac{0.27}{0.72} = \mathbf{\frac{3}{8}} = \mathbf{0.375}.$$

4. d) Suppose a driver insured by this company is not a poor risk. What is the probability that the driver had a traffic accident during 2014?

$$P(\text{Accident} \mid \text{Poor}') = \frac{0.03+0.15}{0.30+0.50} = \frac{0.18}{0.80} = \mathbf{\frac{9}{40}} = \mathbf{0.225}.$$

e) The company announced that it will raise the insurance premiums for the drivers who either are poor risks or had a traffic accident during 2014, or both. What proportion of customers would have their premiums raised?

$P(\text{Poor} \cup \text{Accident}) = P(\text{Poor}) + P(\text{Accident}) - P(\text{Poor} \cap \text{Accident})$

$= 0.20 + 0.28 - 0.10 = \mathbf{0.38}.$

f) Are events {a randomly selected driver is a medium risk} and {a randomly selected driver had a traffic accident during 2014} independent?

$P(\text{Accident} \mid \text{Medium}) = 0.30.$ $P(\text{Accident}) = 0.28.$

Since $P(\text{Accident} \mid \text{Medium}) \neq P(\text{Accident})$, {Medium} and {Accident} are **NOT independent**.

OR

$P(\text{Medium} \mid \text{Accident}) = {}^{0.15}/_{0.28} \approx 0.5357.$ $P(\text{Medium}) = 0.50.$

Since $P(\text{Medium} \mid \text{Accident}) \neq P(\text{Medium})$, {Medium} and {Accident} are **NOT independent**.

OR

$P(\text{Medium} \cap \text{Accident}) = 0.15.$

$P(\text{Medium}) \times P(\text{Accident}) = 0.50 \times 0.28 = 0.14.$

Since $P(\text{Medium} \cap \text{Accident}) \neq P(\text{Medium}) \times P(\text{Accident})$, {Medium} and {Accident} are **NOT independent**.

g) Are events {a randomly selected driver is a medium risk} and {a randomly selected driver had a traffic accident during 2014} mutually exclusive?

Since $P(\text{Medium} \cap \text{Accident}) \neq 0$, {Medium} and {Accident} are **NOT mutually exclusive**.

5. A dashboard warning light is supposed to flash red if a car's oil pressure is too low. On David's aging 2000 Toyota Camry, the probability of the light flashing when it should is 0.97. Due to a known defect, 5% of the time it flashes despite the oil pressure being normal. If there is a 10% chance that the oil pressure really is low, …

$P(\text{ Light} \mid \text{Low }) = 0.97.$ $P(\text{ Light} \mid \text{Low}') = 0.05.$

$P(\text{ Low }) = 0.10.$

a) what is the probability that David needs to be concerned if the warning light comes on?

Bayes' Theorem:

$$P(\text{ Low} \mid \text{Light }) = \frac{P(\text{Low})\times P(\text{Light}\mid\text{Low})}{P(\text{Low})\times P(\text{Light}\mid\text{Low}) + P(\text{Low}')\times P(\text{Light}\mid\text{Low}')}$$

$$= \frac{0.10\times 0.97}{0.10\times 0.97 + 0.90\times 0.05} = \frac{0.097}{0.142} = \mathbf{0.6831}.$$

	Light	Light′	Total
Low	0.097	0.003	0.10
Low′	0.045	0.855	0.90
Total	0.142	0.858	1.00

b) what is the probability that there is no cause for concern if the warning light is not on?

$$P(\text{ Low}' \mid \text{Light}') = \frac{0.855}{0.858} = \mathbf{0.9965}.$$

6. Every morning David uses the CUMTD buses to get to work. After walking to the nearest bus stop he boards the first bus to arrive, which is a "9 Brown" 75% of the time or a "1 Yellow" the other 25% of the time. By riding these two buses for years, David has determined that there is a 10% chance he is late to work if he rides a "9 Brown", and a 50% chance he is late if instead he takes a "1 Yellow".

$P(\text{ Brown }) = 0.75$, $P(\text{ Yellow }) = 0.25$,

$P(\text{ Late} \mid \text{Brown }) = 0.10$, $P(\text{ Late} \mid \text{Yellow }) = 0.50$.

a) What is the probability that David is late to work?

Law of Total Probability:

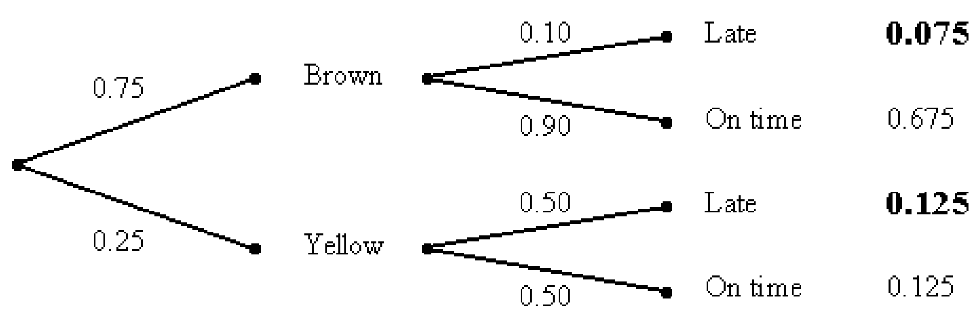
$$P(\text{ Late }) = P(\text{ Brown }) \cdot P(\text{ Late} \mid \text{Brown }) + P(\text{ Yellow }) \cdot P(\text{ Late} \mid \text{Yellow })$$

$$= 0.75 \cdot 0.10 + 0.25 \cdot 0.50 = 0.075 + 0.125 = \mathbf{0.20}.$$

OR

	Late	On Time		
Brown	0.75 · **0.10** 0.075	0.675	**0.75**	P(Late | Brown) = 0.10
Yellow	0.25 · **0.50** 0.125	0.125	**0.25**	P(Late | Yellow) = 0.50
	0.20	0.80	1.00	

OR



0.075+ 0.125= **0.20**.

b) Suppose that David is late to work one morning. What is the probability that he rode the "1 Yellow" that morning?

$$P(\text{ Yellow} \mid \text{Late }) = \frac{P(\text{ Yellow} \cap \text{Late})}{P(\text{ Late})} = \frac{0.125}{0.20} = \mathbf{\frac{5}{8}} = \mathbf{0.625}.$$

c) Suppose David is on time to work. What is the probability that he took the "9 Brown" that morning?

$$P(\text{ Brown} \mid \text{On Time }) = \frac{P(\text{ Brown} \cap \text{On Time})}{P(\text{ On Time})} = \frac{0.675}{0.80} = \mathbf{\frac{27}{32}} = \mathbf{0.84375}.$$

d) David really dislikes riding the "1 Yellow". David is also not a fan of being late to work. What is the probability that David's day has a rough start? (That is, what is the probability that David is late or rode the "1 Yellow", or both?)

$$P(\text{ Yellow} \cup \text{Late }) = P(\text{ Yellow }) + P(\text{ Late }) - P(\text{ Yellow} \cap \text{Late })$$

$$= 0.25 + 0.20 - 0.125 = \mathbf{0.325}.$$

OR

$$P(\text{ Yellow} \cup \text{Late }) = P(\text{ Yellow} \cap \text{Late }) + P(\text{ Yellow} \cap \text{On Time })$$
$$+ P(\text{ Brown} \cap \text{Late })$$

$$= 0.125 + 0.125 + 0.075 = \mathbf{0.325}.$$

OR

$$P(\text{ Yellow} \cup \text{Late }) = 1 - P(\text{ Brown} \cap \text{On Time }) = 1 - 0.675 = \mathbf{0.325}.$$

7. During two-and-a-half years of research, bio-psychologist Onur Güntürkün discovered that when people kiss, they turn their heads to the right roughly twice as often as to the left. (Güntürkün, O. Human behaviour: Adult persistence of head-turning asymmetry. *Nature*, **421**, 711, (2003).)

Suppose the probability that a person would turn his/her head to the right is $^2/_3$, and the probability that a person would turn his/her head to the left is $^1/_3$. A couple is planning a kiss on Valentine's day. Assume that their choice of which way to turn their heads is independent of each other.

a) What is the probability that they would both turn their heads to the right (and kiss)?

Person 1 / Person 2	Right	Left	
Right	$^2/_3 \times {}^2/_3$ $\mathbf{^4/_9}$ kiss	$^1/_3 \times {}^2/_3$ $\mathbf{^2/_9}$ bump noses	$^2/_3$
Left	$^2/_3 \times {}^1/_3$ $\mathbf{^2/_9}$ bump noses	$^1/_3 \times {}^1/_3$ $\mathbf{^1/_9}$ kiss	$^1/_3$
	$^2/_3$	$^1/_3$	1.00

P(both turn their heads to the right) = $\mathbf{^4/_9}$.

b) What is the probability that they would bump noses (that is, choose the opposite direction to turn their heads)?

P(bump noses) = $^2/_9 + {}^2/_9 = \mathbf{^4/_9}$.

Similar arguments with $^3/_4$ and $^1/_4$ predict phenotypic ratios of 9:3:3:1 in F_2 offspring in Mendel's dihybrid crosses.

8. **1.5-16**

An urn contains five balls, one marked WIN and four marked LOSE. You and another player take turns selecting a ball at random from the urn, one at a time. The first person to select the WIN ball is the winner. If you draw first, find the probability that you will win if the sampling is done

a) With replacement.

$$P(1W) + P(1L\ 2L\ \ 1W) + P(1L\ 2L\ \ 1L\ 2L\ \ 1W) + P(1L\ 2L\ \ 1L\ 2L\ \ 1L\ 2L\ \ 1W) + \ldots$$

$$= \frac{1}{5} + \frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5} + \frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5} + \frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5} + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{1}{5}\cdot\left(\frac{16}{25}\right)^k = \frac{\frac{1}{5}}{1-\frac{16}{25}} = \mathbf{\frac{5}{9}}.$$

b) Without replacement.

$$P(1W) + P(1L\ 2L\ \ 1W) + P(1L\ 2L\ \ 1L\ 2L\ \ 1W)$$

$$= \frac{1}{5} + \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{1}{3} + \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \mathbf{\frac{3}{5}}.$$

9. Alex is applying for a job with five companies. At the first company, he is in the final group of four applicants, one of which will be chosen for the position. At two of the five companies, Alex is one of ten candidates; and at the last two companies, he is in an early stage of application in a pool of 25 candidates. Assuming that all companies make their decisions independently of each other, and that Alex is as likely to be chosen as any other applicant, what is the probability of getting at least one job offer?

"at least one" = "either 1st **or** 2nd **or** 3rd **or** 4th **or** 5th" = union.

P(at least one) = 1 – P(none).

"none" = "not 1st **and** not 2nd **and** not 3rd **and** not 4th **and** not 5th".

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = 1 - P(A_1' \cap A_2' \cap A_3' \cap A_4' \cap A_5')$$

since the companies are independent

$$= 1 - P(A_1') \cdot P(A_2') \cdot P(A_3') \cdot P(A_4') \cdot P(A_5')$$

$$= 1 - (0.75) \cdot (0.90) \cdot (0.90) \cdot (0.96) \cdot (0.96) = 1 - 0.56 = \mathbf{0.44}.$$

OR

$$\text{P(at least one job offer)} = 0.25 + (0.75 \cdot 0.10) + (0.75 \cdot 0.90 \cdot 0.10)$$
$$+ (0.75 \cdot 0.90 \cdot 0.90 \cdot 0.04) + (0.75 \cdot 0.90 \cdot 0.90 \cdot 0.96 \cdot 0.04) = \mathbf{0.44}.$$

OR

$$P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5)$$
$$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4) - P(A_1 \cap A_5) - P(A_2 \cap A_3)$$
$$- P(A_2 \cap A_4) - P(A_2 \cap A_5) - P(A_3 \cap A_4) - P(A_3 \cap A_5) - P(A_4 \cap A_5)$$
$$+ P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) + P(A_1 \cap A_2 \cap A_5)$$
$$+ P(A_1 \cap A_3 \cap A_4) + P(A_1 \cap A_3 \cap A_5) + P(A_1 \cap A_4 \cap A_5)$$
$$+ P(A_2 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_5) + P(A_2 \cap A_4 \cap A_5)$$
$$+ P(A_3 \cap A_4 \cap A_5) \quad - P(A_1 \cap A_2 \cap A_3 \cap A_4) - P(A_1 \cap A_2 \cap A_3 \cap A_5)$$
$$- P(A_1 \cap A_2 \cap A_4 \cap A_5) - P(A_1 \cap A_3 \cap A_4 \cap A_5) - P(A_2 \cap A_3 \cap A_4 \cap A_5)$$
$$+ P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \mathbf{\ldots}$$

10. Alex challenged David to a free-throw duel. Alex and David would take turns shooting free throws until someone makes a shot. David makes free throws with probability 0.8. Alex makes free throws with probability 0.4. Assume independence.

A = { Alex makes a free throw }
A′ = { Alex misses a free throw }

D = { David makes a free throw }
D′ = { David misses a free throw }

a) If David shoots first, what is the probability that Alex is the first one to make a free throw?

$$D'A \quad \text{or} \quad D'A'D'A \quad \text{or} \quad D'A'D'A'D'A \quad \text{or} \quad D'A'D'A'D'A'D'A \quad \text{or} \quad \ldots$$

$$0.20 \times 0.40 \ + \ 0.20 \times 0.60 \times 0.20 \times 0.40 \ + \ 0.20 \times 0.60 \times 0.20 \times 0.60 \times 0.20 \times 0.40 \ + \ \ldots$$

$$= \ 0.08 \ + \ 0.12 \times 0.08 \ + \ 0.12^2 \times 0.08 \ + \ \ldots$$

$$= \ \sum_{k=0}^{\infty} 0.08 \cdot 0.12^k \ = \ \frac{0.08}{1 - 0.12} \ = \ \frac{0.08}{0.88} \ = \ \mathbf{\frac{1}{11}}.$$

b) Alex likes to complain. He says that he should shoot first since his success rate is lower. If Alex shoots first, what is the probability that Alex is the first one to make a free throw?

$$A \quad \text{or} \quad A'D'A \quad \text{or} \quad A'D'A'D'A \quad \text{or} \quad A'D'A'D'A'D'A \quad \text{or} \quad \ldots$$

$$0.40 \ + \ 0.60 \times 0.20 \times 0.40 \ + \ 0.60 \times 0.20 \times 0.60 \times 0.20 \times 0.40 \ + \ \ldots$$

$$= \ 0.40 \ + \ 0.12 \times 0.40 \ + \ 0.12^2 \times 0.40 \ + \ \ldots$$

$$= \ \sum_{k=0}^{\infty} 0.40 \cdot 0.12^k \ = \ \frac{0.40}{1 - 0.12} \ = \ \frac{0.40}{0.88} \ = \ \mathbf{\frac{5}{11}}.$$

c) Alex does like to complain. He says that he should have two attempts for each one of David's attempts since his success rate is half of David's. What is the probability that Alex is the first one to make a free throw with these rules? That is, the pattern of attempts is AADAADAAD… instead of ADADAD… in part (b).

$$\text{A} \quad \text{or} \quad \text{A}'\text{A} \quad \text{or} \quad \text{A}'\text{A}'\text{D}'\text{A} \quad \text{or} \quad \text{A}'\text{A}'\text{D}'\text{A}'\text{A}$$

$$\text{or} \quad \text{A}'\text{A}'\text{D}'\text{A}'\text{A}'\text{D}'\text{A} \quad \text{or} \quad \text{A}'\text{A}'\text{D}'\text{A}'\text{A}'\text{D}'\text{A}'\text{A}$$

$$\text{or} \quad \text{A}'\text{A}'\text{D}'\text{A}'\text{A}'\text{D}'\text{A}'\text{A}'\text{D}'\text{A} \quad \text{or} \quad \text{A}'\text{A}'\text{D}'\text{A}'\text{A}'\text{D}'\text{A}'\text{A}'\text{D}'\text{A}'\text{A}$$

$$\text{or} \quad \ldots$$

$$0.40 + 0.60 \times 0.40 + 0.60 \times 0.60 \times 0.20 \times 0.40 + 0.60 \times 0.60 \times 0.20 \times 0.60 \times 0.40$$

$$+ 0.60 \times 0.60 \times 0.20 \times 0.60 \times 0.60 \times 0.20 \times 0.40$$

$$+ 0.60 \times 0.60 \times 0.20 \times 0.60 \times 0.60 \times 0.20 \times 0.60 \times 0.40$$

$$+ \ldots$$

$$= 0.64 + 0.072 \times 0.64 + 0.072^2 \times 0.64 + \ldots$$

$$= \sum_{k=0}^{\infty} 0.64 \cdot 0.072^k = \frac{0.64}{1 - 0.072} = \frac{0.64}{0.928} = \mathbf{\frac{20}{29}}.$$