STAT 400
UIUC

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Practice Problems #4

SOLUTIONS

The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1 – 2. When Stéphane plays chess against his favorite computer program, he wins with probability 0.60, loses with probability 0.10, and 30% of the games result is a draw. Assume independence.

1. a) Find the probability that Stéphane's first win happens when he plays his third game.

Geometric, $p = 0.60$. $P(X = 3) = 0.40^2 \cdot 0.60 = \mathbf{0.096}$.

b) Find the probability that Stéphane's fifth win happens when he plays his eighth game.

Negative Binomial, $p = 0.60$, $r = 5$.

$P(X = 8) = {}_7C_4 \cdot 0.60^5 \cdot 0.40^3 \approx \mathbf{0.17418}$.

c) Find the probability that Stéphane wins 7 games, if he plays 10 games.

Binomial, $n = 10$, $p = 0.60$.

$P(X = 7) = {}_{10}C_7 \cdot 0.60^7 \cdot 0.40^3 \approx \mathbf{0.21499}$.

2. Stéphane plays 12 games.

c) Find the probability that he wins 5 games, loses 3 games, and draws 4 games.

Multinomial.

$$\frac{12!}{5!\cdot 3!\cdot 4!}\cdot 0.60^{5}\cdot 0.10^{3}\cdot 0.30^{4} \approx \mathbf{0.017460}.$$

d) Find the probability that he wins 7 games, and draws 5 games.

Multinomial.

$$\frac{12!}{7!\cdot 0!\cdot 5!}\cdot 0.60^{7}\cdot 0.10^{0}\cdot 0.30^{5} \approx \mathbf{0.053875}.$$

e) Find the probability that Stéphane wins at least 8 games.

Binomial, $n = 12$, $p = 0.60$.

$$P(X \geq 8) = {}_{12}C_{8}\cdot 0.60^{8}\cdot 0.40^{4} + {}_{12}C_{9}\cdot 0.60^{9}\cdot 0.40^{3} + {}_{12}C_{10}\cdot 0.60^{10}\cdot 0.40^{2}$$

$$+ {}_{12}C_{11}\cdot 0.60^{11}\cdot 0.40^{1} + {}_{12}C_{12}\cdot 0.60^{12}\cdot 0.40^{0} \approx \mathbf{0.438178}.$$

3. a) Alex takes a multiple choice quiz in his Anthropology 100 class. The quiz has 10 questions, each has 4 possible answers, only one of which is correct. Alex did not study for the quiz, so he guesses independently on every question. What is the probability that Alex answers exactly 2 questions correctly?

X = number of questions Alex answers correctly.

Binomial, $n = 10$, $p = {}^{1}/_{4} = 0.25$.

$\mathrm{P}(\mathrm{X} = 2) = {}_{10}C_2 \cdot (0.25)^2 \cdot (0.75)^8 \approx \mathbf{0.28157}$.

b) Alex takes a quiz in his Anthropology 100 class. The quiz consists of 10 questions, the first 4 are True-False, the last 6 are multiple choice questions with 4 possible answers each, only one of which is correct. Alex did not study for the quiz, so he guesses independently on each question. Find the probability that he answers exactly 2 questions correctly.

Let X = the number of True-False questions answered correctly,

Y = the number of multiple choice questions answered correctly.

X has Binomial distribution, $n_1 = 4$, $p_1 = 0.50$.

Y has Binomial distribution, $n_2 = 6$, $p_2 = 0.25$.

$$\mathrm{P}(\mathrm{X} + \mathrm{Y} = 2) = \mathrm{P}(\mathrm{X} = 0 \cap \mathrm{Y} = 2) + \mathrm{P}(\mathrm{X} = 1 \cap \mathrm{Y} = 1) + \mathrm{P}(\mathrm{X} = 2 \cap \mathrm{Y} = 0)$$

$$= \mathrm{P}(\mathrm{X} = 0) \times \mathrm{P}(\mathrm{Y} = 2) + \mathrm{P}(\mathrm{X} = 1) \times \mathrm{P}(\mathrm{Y} = 1) + \mathrm{P}(\mathrm{X} = 2) \times \mathrm{P}(\mathrm{Y} = 0)$$

$$= {}_4C_0 \cdot 0.50^0 \cdot 0.50^4 \times {}_6C_2 \cdot 0.25^2 \cdot 0.75^4$$

$$+ {}_4C_1 \cdot 0.50^1 \cdot 0.50^3 \times {}_6C_1 \cdot 0.25^1 \cdot 0.75^5$$

$$+ {}_4C_2 \cdot 0.50^2 \cdot 0.50^2 \times {}_6C_0 \cdot 0.25^0 \cdot 0.75^6$$

$$\approx 0.01854 + 0.08899 + 0.06674 = \mathbf{0.17427}.$$

4. When correctly adjusted, a machine that makes widgets operates with a 5% defective rate. However, there is a 10% chance that a disgruntled employee kicks the machine, in which case the defective rate jumps up to 30%.

a) Suppose that a widget made by this machine is selected at random and is found to be defective. What is the probability that the machine had been kicked?

$$P(D) = 0.90 \times 0.05 + 0.10 \times 0.30 = 0.075.$$

$$P(K \mid D) = \frac{0.10 \times 0.30}{0.90 \times 0.05 + 0.10 \times 0.30} = \frac{0.030}{0.075} = \mathbf{0.40}.$$

b) A random sample of 20 widgets was examined, 4 widgets out of these 20 are found to be defective. What is the probability that the machine had been kicked?

Hint: What is the probability of finding 4 defective widgets in a sample of 20, if (given) the machine has been kicked?
What is the probability of finding 4 defective widgets in a sample of 20, if (given) the machine has not been kicked?

$$P(X = 4 \mid K') = \binom{20}{4} (0.05)^4 (0.95)^{16} = 0.0133,$$

$$P(X = 4 \mid K) = \binom{20}{4} (0.30)^4 (0.70)^{16} = 0.1304.$$

$$P(K \mid X = 4) = \frac{0.10 \times 0.1304}{0.90 \times 0.0133 + 0.10 \times 0.1304} = \frac{0.01304}{0.02501} \approx \mathbf{0.52}.$$

5. Find the probability $P(\mu - \sigma < X < \mu + \sigma)$ if X has …

a) … a Binomial distribution with $n = 15$ and $p = \frac{1}{3}$.

Binomial, $n = 15$, $p = \frac{1}{3}$ $\qquad \mu = n\,p = 5$, $\qquad \sigma^2 = n\,p\,(1-p) = \frac{10}{3}$,

$$\sigma = \sqrt{\frac{10}{3}} \approx 1.82574.$$

$$P(\mu - \sigma < X < \mu + \sigma) = P(3.17426 < X < 6.82574) = P(4 \le X \le 6)$$

$$= {}_{15}C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{11} + {}_{15}C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^{10} + {}_{15}C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^9$$

$$\approx 0.19482 + 0.21431 + 0.17859 = \mathbf{0.58772}.$$

b) … a Geometric distribution with $p = \frac{1}{5}$.

Geometric, $p = \frac{1}{5}$ $\qquad \mu = \frac{1}{p} = 5$, $\qquad \sigma^2 = \frac{1-p}{p^2} = 20$,

$$\sigma = \sqrt{20} \approx 4.472.$$

$$P(\mu - \sigma < X < \mu + \sigma) = P(0.528 < X < 9.472) = P(1 \le X \le 9)$$

For Geometric(p), $\qquad P(X > a) = (1-p)^a$, $\quad a = 0, 1, 2, \ldots$.

$$P(1 \le X \le 9) = 1 - P(X > 9) = 1 - \left(\frac{4}{5}\right)^9 \approx \mathbf{0.86578}.$$

OR

$$P(1 \le X \le 9) = \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^1\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)$$

$$+ \left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^5\left(\frac{1}{5}\right)$$

$$+ \left(\frac{4}{5}\right)^6\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^7\left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^8\left(\frac{1}{5}\right) \approx \mathbf{0.86578}.$$

c) … a Poisson distribution with $\lambda = 5$.

Poisson, $\lambda = 5$ $\Rightarrow$ $\mu = \lambda = 5$, $\sigma^2 = \lambda = 5$, $\sigma = \sqrt{5} \approx 2.236$.

$$P(\mu - \sigma < X < \mu + \sigma) = P(2.764 < X < 7.236) = P(3 \le X \le 7)$$

$$= \frac{5^3 e^{-5}}{3!} + \frac{5^4 e^{-5}}{4!} + \frac{5^5 e^{-5}}{5!} + \frac{5^6 e^{-5}}{6!} + \frac{5^7 e^{-5}}{7!}$$

$$\approx 0.1404 + 0.1755 + 0.1755 + 0.1462 + 0.1044 = \mathbf{0.7420}.$$

OR

$$P(3 \le X \le 7) = \text{CDF @ 7} - \text{CDF @ 2} = 0.867 - 0.125 = \mathbf{0.742}.$$

6. Poor Milhouse is hopelessly in love with Lisa. Unfortunately for Milhouse, Lisa does not feel the same way. However, Milhouse remains hopeful, since on any given day independently there is a 5% chance that Lisa smiles at him. (Assume a month has 30 days for this problem.)

a) What is the probability that Milhouse goes longer than a month without a smile from Lisa?

Waiting for the first "success" $\Rightarrow$ Geometric, $p = 0.05$.

For Geometric(p), $P(X > a) = (1-p)^a$, $a = 0, 1, 2, \ldots$.

$P(X > 30) = 0.95^{30} \approx \mathbf{0.21464}$.

OR

Counting "successes" $\Rightarrow$ Binomial, $n = 30$, $p = 0.05$.

$P(X = 0) = {}_{30}C_0 \cdot 0.05^0 \cdot 0.95^{30} = 0.95^{30} \approx \mathbf{0.21464}$.

b) What is the probability that Milhouse waits no more than two months to see a smile from Lisa?

Waiting for the first "success" $\Rightarrow$ Geometric, $p = 0.05$.

For Geometric(p), $P(X > a) = (1-p)^a$, $a = 0, 1, 2, \ldots$.

$P(X \le 60) = 1 - P(X > 60) = 1 - 0.95^{60} \approx \mathbf{0.95393}$.

OR

Counting "successes" $\Rightarrow$ Binomial, $n = 60$, $p = 0.05$.

$P(X \ge 1) = 1 - P(X = 0) = 1 - {}_{60}C_0 \cdot 0.05^0 \cdot 0.95^{60} = 1 - 0.95^{60} \approx \mathbf{0.95393}$.

c) What is the probability that Lisa smiles at Milhouse on exactly three days in a month?

Counting "successes" $\Rightarrow$ Binomial, $n = 30$, $p = 0.05$.

$P(X = 3) = {}_{30}C_3 \cdot 0.05^3 \cdot 0.95^{27} = 4060 \cdot 0.05^3 \cdot 0.95^{27} \approx \mathbf{0.12705}$.

7. An advertising company has 12 men and 8 women. Suppose the company needs to select a team of 5 members to work on a commercial for the new hybrid car, Hyper Geo Metro 2015 (☺).

a) If the members of the team are selected at random, what is the probability that 3 men and 2 women will be selected?

$$\frac{\binom{12}{3}\cdot\binom{8}{2}}{\binom{20}{5}} = \frac{220\cdot 28}{15504} \approx \mathbf{0.3973}.$$

b) What is the probability that men will constitute a majority in the team?

$$\frac{\binom{12}{3}\cdot\binom{8}{2}}{\binom{20}{5}} + \frac{\binom{12}{4}\cdot\binom{8}{1}}{\binom{20}{5}} + \frac{\binom{12}{5}\cdot\binom{8}{0}}{\binom{20}{5}} = \frac{220\cdot 28}{15504} + \frac{495\cdot 8}{15504} + \frac{792\cdot 1}{15504}$$

$$\approx 0.3973 + 0.2554 + 0.0511 = \mathbf{0.7038}.$$

8. Let X equal the number of people selected at random that you must ask in order to find someone with the same birthday as yours. Assume that each day of the year is equally likely, and ignore February 29.

a) Give the values of the mean and the variance of X.

Geometric, $p = \dfrac{1}{365}$.

$$f(k) = \left(\frac{364}{365}\right)^{k-1}\left(\frac{1}{365}\right), \quad k = 1, 2, 3, \ldots .$$

$\mu = \dfrac{1}{p} = \mathbf{365}$,

$$\sigma^2 = \frac{1-p}{p^2} = 364 \cdot 365 = \mathbf{132{,}860}.$$

b) Find $P(X > 400)$.

For Geometric(p), $\quad P(X > a) = (1-p)^a, \quad a = 0, 1, 2, \ldots .$

$$\Rightarrow \quad P(X > 400) = \left(\frac{364}{365}\right)^{400} = \mathbf{0.33374}.$$

c) Find $P(X < 300)$.

For Geometric(p), $\quad P(X > a) = (1-p)^a, \quad a = 0, 1, 2, \ldots .$

$$\Rightarrow \quad P(X < 300) = P(X \le 299) = 1 - P(X > 299) = 1 - \left(\frac{364}{365}\right)^{299} = \mathbf{0.5597}.$$

9. Alex makes mistakes in class according to Poisson process with an average rate of 1.2 mistakes per class.

a) What is the probability that Alex makes at least 3 mistakes during one class?

1 class $\Rightarrow$ $\lambda = 1.2.$

Need $P(X \geq 3) = ?$ Poisson distribution: $P(X = x) = \dfrac{\lambda^{x} \cdot e^{-\lambda}}{x!}$

$$P(X \geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[\frac{1.2^{0} \cdot e^{-1.2}}{0!} + \frac{1.2^{1} \cdot e^{-1.2}}{1!} + \frac{1.2^{2} \cdot e^{-1.2}}{2!}\right]$$

$$= 1 - [0.3012 + 0.3614 + 0.2169] = 1 - 0.8795 = \mathbf{0.1205}.$$

b) What is the probability that Alex makes exactly 10 mistakes during two weeks of classes (that is, during 6 classes, since Alex teaches a MWF lecture)?

6 classes $\Rightarrow$ $\lambda = 1.2 \cdot 6 = 7.2.$ $\quad P(X = 10) = \dfrac{7.2^{10} \cdot e^{-7.2}}{10!} = \mathbf{0.0770}.$

c) What is the probability that Alex has exactly two mistake-free classes in two weeks (that is, during 6 classes)?

Let Y = the number of mistake-free classes in one week (6 classes).

Then Y has Binomial distribution, $n = 6$, $p = \dfrac{1.2^{0} \cdot e^{-1.2}}{0!} = 0.3012.$

$$P(Y = 2) = {}_{6}C_{2} \cdot (0.3012)^{2} \cdot (1 - 0.3012)^{4} = \mathbf{0.3245}.$$

10. In a casino game chuck-a-luck, three unbiased six-sided dice are rolled. One possible bet is \$1 on fives, and the payoff is equal to \$1 for each five on that roll. In addition, the dollar bet is returned if at least one five is rolled. The dollar that was bet is lost only if no fives are rolled. Let X denote the payoff for this game. Then X can equal – 1, 1, 2, or 3.

a) Determine the p.m.f. $f(x)$.

$$f(-1) = \mathrm{P}(\text{no fives}) = {}_3C_0 \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^3 = \frac{125}{216}.$$

$$f(1) = \mathrm{P}(\text{one five}) = {}_3C_1 \cdot \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^2 = \frac{75}{216}.$$

$$f(2) = \mathrm{P}(\text{two fives}) = {}_3C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^1 = \frac{15}{216}.$$

$$f(3) = \mathrm{P}(\text{three fives}) = {}_3C_3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^0 = \frac{1}{216}.$$

b) Calculate μ and σ^2.

x	$f(x)$	$x \times f(x)$	$x^2 \times f(x)$
-1	$\frac{125}{216}$	$-\frac{125}{216}$	$\frac{125}{216}$
1	$\frac{75}{216}$	$\frac{75}{216}$	$\frac{75}{216}$
2	$\frac{15}{216}$	$\frac{30}{216}$	$\frac{60}{216}$
3	$\frac{1}{216}$	$\frac{3}{216}$	$\frac{9}{216}$
		$-\frac{17}{216}$	$\frac{269}{216}$

$$\mu = \mathrm{E}(\mathrm{X}) = \mathbf{-\frac{17}{216}} \approx \mathbf{-0.0787037}.$$

$$\sigma^2 = \mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = \frac{269}{216} - \left(-\frac{17}{216}\right)^2 \approx \mathbf{1.239176}.$$