STAT 400
UIUC

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Practice Problems #5

SOLUTIONS

The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1. Each week, Stéphane needs to prepare 4 exercises for the following week's homework assignment. The number of problems he creates in a week follows a Poisson distribution with mean 6.

a) What is the probability that Stéphane manages to create enough exercises for the following week's homework?

$$P(X \ge 4) = 1 - \frac{6^0 \cdot e^{-6}}{0!} - \frac{6^1 \cdot e^{-6}}{1!} - \frac{6^2 \cdot e^{-6}}{2!} - \frac{6^3 \cdot e^{-6}}{3!}$$

$$= 1 - 0.0025 - 0.0149 - 0.0446 - 0.0892 = 1 - 0.1512 = \mathbf{0.8488}.$$

b) Unfortunately, each week there is a 40% chance that a visiting scholar from Switzerland arrives and burdens Stéphane with research questions all week. During these weeks he only writes an average of 3 exercises. If Stéphane fails to write 4 exercises one week, what is the probably that he received a visiting scholar that week?

$P(\text{scholar}) = 0.40.$ $\qquad$ $P(X < 4 \mid \text{scholar}') = 0.1512.$

$$P(X < 4 \mid \text{scholar}) = \frac{3^0 \cdot e^{-3}}{0!} + \frac{3^1 \cdot e^{-3}}{1!} + \frac{3^2 \cdot e^{-3}}{2!} + \frac{3^3 \cdot e^{-3}}{3!}$$

$$= 0.0498 + 0.1494 + 0.2240 + 0.2240 = 0.6472.$$

$$P(\text{scholar} \mid X < 4) = \frac{P(\text{scholar}) \times P(X < 4 \mid \text{scholar})}{P(\text{scholar}) \times P(X < 4 \mid \text{scholar}) + P(\text{scholar}') \times P(X < 4 \mid \text{scholar}')}$$

$$= \frac{0.40 \cdot 0.6472}{0.40 \cdot 0.6472 + 0.60 \cdot 0.1512} = \frac{0.25888}{0.3496} = \mathbf{0.7405}.$$

c) The last week of the semester, Stéphane decides to “reward” the students by no longer limiting himself to 4 exercises, and instead assigning every exercise he writes. If a student with a 60% chance of correctly answering an exercise is expected to answer 3 correctly, what is the probably that Stéphane did not have a visitor that week?

$$n\,p \;=\; n \cdot 0.60 \;=\; 3. \qquad \Rightarrow \qquad n \;=\; 5.$$

$$\text{P}(\,\text{X} = 5 \,|\, \text{scholar}\,) \;=\; \frac{3^{\,5} \cdot e^{\,-3}}{5!} \;=\; 0.1008.$$

$$\text{P}(\,\text{X} = 5 \,|\, \text{scholar}'\,) \;=\; \frac{6^{\,5} \cdot e^{\,-6}}{5!} \;=\; 0.1606.$$

$$\text{P}(\,\text{scholar}' \,|\, \text{X} = 5\,) \;=\; \frac{\text{P}(\text{scholar}') \times \text{P}(\text{X} = 5 | \text{scholar}')}{\text{P}(\text{scholar}) \times \text{P}(\text{X} = 5 | \text{scholar}) + \text{P}(\text{scholar}') \times \text{P}(\text{X} = 5 | \text{scholar}')}$$

$$= \frac{0.60 \cdot 0.1606}{0.40 \cdot 0.1008 + 0.60 \cdot 0.1606} \;=\; \frac{0.09636}{0.13668} \;=\; \mathbf{0.7050}.$$

2. Alex uses a copy machine to make 225 copies of the exam. Suppose that for each copy of the exam the stapler independently malfunctions with probability 0.02.

a) Find the probability that the stapler would malfunction exactly 3 times.

Let X = number of times the stapler malfunctions.

Binomial distribution, $n = 225$, $p = 0.02$.

$$P(X = 3) = {}_{225}C_3 \cdot 0.02^3 \cdot 0.98^{222} = \mathbf{0.16899}.$$

b) Use Poisson approximation to find the probability that the stapler would malfunction exactly 3 times.

Poisson approximation: $\lambda = n \cdot p = 225 \cdot 0.02 = 4.5$.

$$P(X = 3) = \frac{4.5^3 \cdot e^{-4.5}}{3!} = \mathbf{0.16872}.$$

c) Find the probability that the stapler would malfunction at least 3 times.

$$P(X \ge 3) = 1 - {}_{225}C_0 \cdot 0.02^0 \cdot 0.98^{225} - {}_{225}C_1 \cdot 0.02^1 \cdot 0.98^{224} - {}_{225}C_2 \cdot 0.02^2 \cdot 0.98^{223}$$

$$= 1 - 0.01061 - 0.04874 - 0.11140 = \mathbf{0.82925}.$$

d) Use Poisson approximation to find the probability that the stapler would malfunction at least 3 times.

$$P(X \ge 3) = 1 - \frac{4.5^0 \cdot e^{-4.5}}{0!} - \frac{4.5^1 \cdot e^{-4.5}}{1!} - \frac{4.5^2 \cdot e^{-4.5}}{2!}$$

$$= 1 - 0.01111 - 0.04999 - 0.11248 = \mathbf{0.82642}.$$

3. Let X be a discrete random variable with p.m.f.

$$f(k) = \frac{c}{a^k}, \quad k = 2, 3, 4, 5, 6, \dots, \qquad \text{where } c = a(a-1).$$

Recall (Homework #1 Problem 7): this a valid probability distribution.

a) Find the moment-generating function of X, $M_X(t)$. For which values of t does it exist?

$$M_X(t) = \sum_{\text{all } x} e^{tx} \cdot f(x) = \sum_{k=2}^{\infty} e^{tk} \cdot \frac{c}{a^k} = c \cdot \sum_{k=2}^{\infty} \left(\frac{e^t}{a}\right)^k$$

Geometric series

$$= \frac{\text{first term}}{1-\text{base}} = \frac{c\left(\frac{e^t}{a}\right)^2}{1-\left(\frac{e^t}{a}\right)} = \frac{(a-1)e^{2t}}{a-e^t},$$

$$\text{if} \quad \frac{e^t}{a} < 1 \quad \Leftrightarrow \quad t < \ln a.$$

b) Use $M_X(t)$ to find $\mu_X = E(X)$. ☺ *We already know what the answer is.* ☺ *Therefore, show ALL work.* ☹ *No credit will be given without supporting work.* ☹

$$M'_X(t) = \frac{2(a-1)e^{2t}\left(a-e^t\right)-(a-1)e^{2t}\left(-e^t\right)}{\left(a-e^t\right)^2}$$

$$= \frac{2a(a-1)e^{2t}-(a-1)e^{3t}}{\left(a-e^t\right)^2}, \quad t < \ln a.$$

$$E(X) = M'_X(0) = \frac{2a(a-1)-(a-1)}{(a-1)^2} = \frac{2a-1}{a-1}.$$

4. Let X be a discrete random variable with p.m.f.

$$f(k) = c\,\frac{2^k}{k!}, \quad k = 2, 3, 4, 5, 6, \ldots, \qquad \text{where } c = \frac{1}{e^2 - 3}.$$

Recall (Homework #1 Problem 8): this a valid probability distribution.

a) Find the moment-generating function of X, $M_X(t)$. For which values of t does it exist?

$$M_X(t) = \sum_{\text{all } x} e^{tx} \cdot p(x) = \sum_{k=2}^{\infty} e^{tk} \cdot \frac{1}{e^2 - 3}\,\frac{2^k}{k!} = \frac{1}{e^2 - 3} \sum_{k=2}^{\infty} \frac{\left(2e^t\right)^k}{k!}$$

$$= \frac{1}{e^2 - 3}\left(e^{2e^t} - 1 - 2e^t\right), \qquad t \in \mathbf{R}.$$

b) Use $M_X(t)$ to find $\mu_X = E(X)$. ☺ *We already know what the answer is.* ☺ *Therefore, show ALL work.* ☹ *No credit will be given without supporting work.* ☹

$$E(X) = M_X'(0) = \frac{1}{e^2 - 3}\left(e^{2e^t} \cdot 2e^t - 2e^t\right)\bigg|_{t=0} = \frac{2\left(e^2 - 1\right)}{e^2 - 3}.$$

5. How much wood would a woodchuck chuck if a woodchuck could chuck wood? Let X denote the amount of wood a woodchuck would chuck per day (in cubic meters) if a woodchuck could chuck wood. Suppose the moment-generating function of X is

$$M_X(t) = 0.1\,e^{8t} + 0.2\,e^{7t} + 0.3\,e^{6t} + 0.4\,e^{5t}.$$

Find the average amount of wood a woodchuck would chuck per day, $E(X)$, and the variance $\operatorname{Var}(X)$.

$$M_X'(t) = 0.8\,e^{8t} + 1.4\,e^{7t} + 1.8\,e^{6t} + 2.0\,e^{5t}.$$

$$E(X) = M_X'(0) = 0.8 + 1.4 + 1.8 + 2.0 = \mathbf{6}.$$

$$M_X''(t) = 6.4\,e^{8t} + 9.8\,e^{7t} + 10.8\,e^{6t} + 10.0\,e^{5t}.$$

$$E(X^2) = M_X''(0) = 6.4 + 9.8 + 10.8 + 10.0 = 37.$$

$$\operatorname{Var}(X) = E(X^2) - [E(X)]^2 = 37 - 6^2 = \mathbf{1}.$$

OR

$$M_X(t) = 0.1\,e^{8t} + 0.2\,e^{7t} + 0.3\,e^{6t} + 0.4\,e^{5t}. \quad \Rightarrow$$

x	$f(x)$
5	0.4
6	0.3
7	0.2
8	0.1

x	$f(x)$	$x \cdot f(x)$	$x^2 \cdot f(x)$	$(x-\mu)^2 \cdot f(x)$
5	0.4	2.0	10.0	0.4
6	0.3	1.8	10.8	0.0
7	0.2	1.4	9.8	0.2
8	0.1	0.8	6.4	0.4
		6.0	37.0	**1.0**

$$\mu = E(X) = \sum x \cdot f(x) = \mathbf{6.0}. \qquad \operatorname{Var}(X) = \sum (x-\mu)^2 \cdot f(x) = \mathbf{1.0}.$$

OR $$\operatorname{Var}(X) = \sum x^2 \cdot f(x) - \mu^2 = 37.0 - 6.0^2 = \mathbf{1.0}.$$

6. Let X be a continuous random variable with the probability density function

$$f(x) = C\,x, \qquad 5 \le x \le 11, \qquad \text{zero otherwise.}$$

a) Find the value of C that would make $f(x)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{5}^{11} C\,x\,dx = C\,\frac{x^2}{2}\Bigg|_5^{11} = C\,\frac{121-25}{2} = 48\,C.$$

$$\Rightarrow \qquad C = \frac{\mathbf{1}}{\mathbf{48}}.$$

$$f(x) = \frac{x}{48}, \qquad 5 \le x \le 11, \qquad \text{zero otherwise.}$$

b) Find the probability $\mathrm{P}(\mathrm{X} > 7)$.

$$\mathrm{P}(\mathrm{X} > 7) = \int_{7}^{11} \frac{x}{48}\,dx = \frac{x^2}{96}\Bigg|_7^{11} = \frac{72}{96} = \frac{\mathbf{3}}{\mathbf{4}} = \mathbf{0.75}.$$

c) Find the mean of the probability distribution of X.

$$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{5}^{11} x \cdot \frac{x}{48}\,dx = \frac{x^3}{144}\Bigg|_5^{11} = \frac{1206}{144} = \frac{\mathbf{67}}{\mathbf{8}} = \mathbf{8.375}.$$

d) Find the median of the probability distribution of X.

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = \int_{-\infty}^{x} f(y)\,dy = \int_{5}^{x} \frac{y}{48}\,dy = \frac{y^2}{96}\Bigg|_5^{x} = \frac{x^2 - 25}{96}, \qquad 5 \le x < 11.$$

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = 0, \qquad x < 5.$$

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x) = 1, \qquad x \ge 11.$$

$$\mathrm{F}(m) = \frac{1}{2}. \qquad \frac{m^2 - 25}{96} = \frac{1}{2}.$$

$$m^2 = \frac{96}{2} + 25 = 73. \qquad m = \sqrt{\mathbf{73}} \approx 8.544.$$

7 – 8. Suppose the length of time X (in hours) it takes for pizza to be delivered by *Momma Leona's Pizza* has the probability density function

$$f(x) = \begin{cases} c\left(x^2 - x^3\right), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

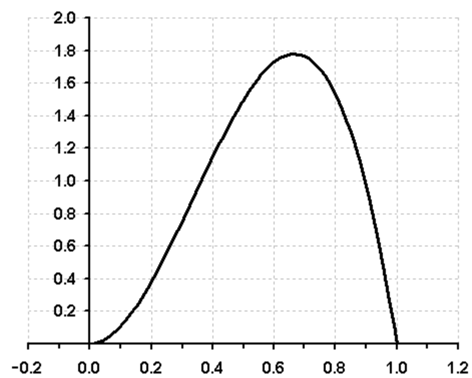
7. a) Find the value of c that makes $f(x)$ a valid probability density function.

Must have $\int_{-\infty}^{\infty} f(x)dx = 1$. Then

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 c\left(x^2 - x^3\right)dx$$

$$= \left[\frac{c}{3}\cdot x^3 - \frac{c}{4}\cdot x^4 \right]\Bigg|_0^1 = \frac{c}{12}.$$

$\Rightarrow \quad {}^{c}/_{12} = 1. \quad \Rightarrow \quad c =$ **12**.



b) Find $\mu = \mathrm{E}(\mathrm{X})$, the average delivery time.

$$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x\cdot f(x)dx = \int_0^1 12\left(x^3 - x^4\right)dx = \left(3x^4 - \frac{12}{5}x^5\right)\Bigg|_0^1 = \mathbf{0.6\ hours}$$

= **36 minutes**.

c) Find $\sigma = \mathrm{SD}(\mathrm{X})$.

$$\mathrm{Var}(\mathrm{X}) = \left(\int_{-\infty}^{\infty} x^2\cdot f(x)dx\right) - [\mathrm{E}(\mathrm{X})]^2 = \int_0^1 12\left(x^4 - x^5\right)dx - [0.6]^2$$

$$= \left(\frac{12}{5}x^5 - 2x^6\right)\Bigg|_0^1 - [0.6]^2 = 0.40 - 0.36 = 0.04.$$

$\mathrm{SD}(\mathrm{X}) = \sqrt{0.04} =$ **0.2 hours** = **12 minutes**.

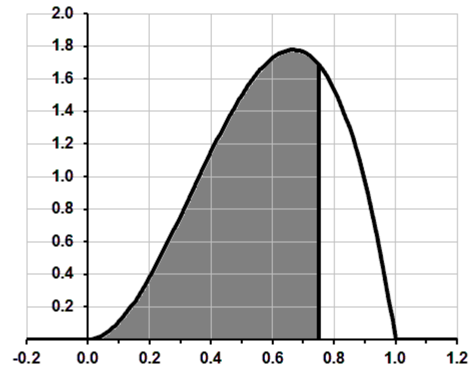
8. d) Find the probability that it will take less than 45 minutes for a pizza to be delivered.

$$P(X < 0.75) = \int_0^{0.75} 12\left(x^2 - x^3\right)dx$$

$$= \left(4 \cdot x^3 - 3 \cdot x^4\right)\Big|_0^{0.75}$$

$$= \left[\frac{27}{16} - \frac{243}{256}\right] - 0$$

$$= \mathbf{\frac{189}{256}} \approx 0.73828.$$



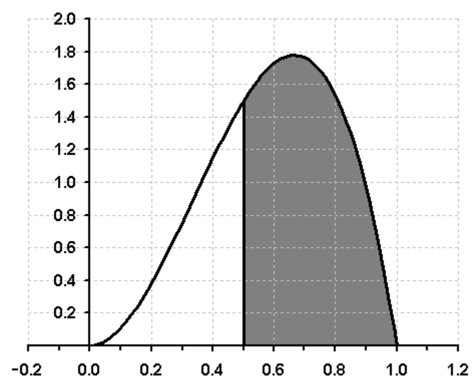
e) Find the probability that it will take more than 30 minutes for a pizza to be delivered.

$$P(X > 0.50) = \int_{0.50}^{1} 12\left(x^2 - x^3\right)dx$$

$$= \left(4 \cdot x^3 - 3 \cdot x^4\right)\Big|_{0.50}^{1}$$

$$= [4 - 3] - \left[\frac{4}{8} - \frac{3}{16}\right]$$

$$= \mathbf{\frac{11}{16}} = \mathbf{0.6875}.$$



f) If 7 independent deliveries are made on a particular day, what is the probability that exactly 5 of them took more than 30 minutes.

$${}_7C_5 \cdot \left(\frac{11}{16}\right)^5 \cdot \left(\frac{5}{16}\right)^2 = \mathbf{0.31498}.$$

9 – 10. Suppose a random variable X has the following probability density function:

$$f(x) = \frac{1}{x}, \qquad 1 < x < C, \qquad \text{zero otherwise.}$$

9. a) What must the value of C be so that $f(x)$ is a probability density function?

For $f(x)$ to be a probability density function, we must have:

1) $f(x) \ge 0$, 2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{1}^{C} \frac{1}{x}\,dx = \ln C - \ln 1 = \ln C.$$

Therefore, $C = \boldsymbol{e}$.

b) Find $\mu = \mathrm{E}(\mathrm{X})$.

$$\mu_{\mathrm{X}} = \mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{1}^{e} x \cdot \frac{1}{x}\,dx = \int_{1}^{e} 1\,dx = \boldsymbol{e-1} \approx 1.718282.$$

c) Find $\sigma = \mathrm{SD}(\mathrm{X})$.

$$\mathrm{E}(\mathrm{X}^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_{1}^{e} x^2 \cdot \frac{1}{x}\,dx = \int_{1}^{e} x\,dx = \frac{e^2 - 1}{2} \approx 3.194528.$$

$$\sigma_{\mathrm{X}}^2 = \mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = \frac{-e^2 + 4e - 3}{2} \approx 0.242036.$$

$$\sigma_{\mathrm{X}} = \mathrm{SD}(\mathrm{X}) = \sqrt{0.242036} \approx 0.491971.$$

10. d) Find $P(X < 2)$.

$$P(X < 2) = \int_{-\infty}^{2} f(x)\,dx = \int_{1}^{2} \frac{1}{x}\,dx = \ln 2 - \ln 1 = \mathbf{\ln 2}.$$

e) Find $P(X < 3)$.

$$P(X < 3) = \int_{-\infty}^{3} f(x)\,dx = \int_{1}^{e} \frac{1}{x}\,dx = \ln e - \ln 1 = \mathbf{1}.$$

f) Find the median of the probability distribution of X.

$$F(x) = \int_{1}^{x} \frac{1}{y}\,dy = \ln x - \ln 1 = \ln x, \qquad 1 < x < e.$$

$(100p)$th percentile: $\quad F(\pi_p) = p.$

$$\ln \pi_p = p. \qquad \pi_p = e^{p}.$$

$$\text{median} = \text{50th percentile} = e^{\frac{1}{2}} = \sqrt{e} \approx 1.648721.$$

g) Find the 25th percentile of the probability distribution of X.

$$\text{25th percentile} = e^{\frac{1}{4}} = \sqrt[4]{e} \approx 1.284025.$$