STAT 400
UIUC

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Practice Problems #6

SOLUTIONS

The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1. Models of the pricing of stock options often make the assumption of a normal distribution. An analyst believes that the price of an *Initech* stock option varies from day to day according to normal distribution with mean $9.22 and unknown standard deviation.

a) The analyst also believes that 77% of the time the price of the option is greater than $7.00. Find the standard deviation of the price of the option.

$\mu = 9.22$, $\sigma = ?$ Know $P(X > 7.00) = 0.77$.

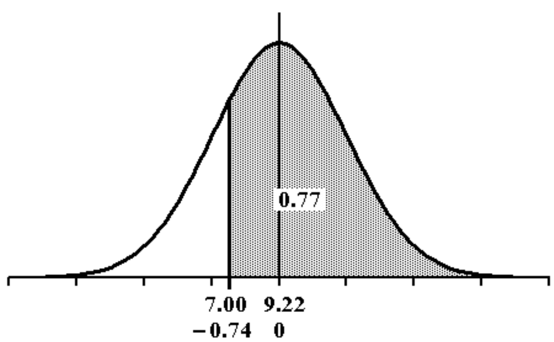
① Find z such that $P(Z > z) = 0.77$.

$z = \mathbf{-0.74}$.

② $x = \mu + \sigma \cdot z$.

$7.00 = 9.22 + \sigma \cdot (-0.74)$.

$\sigma = \mathbf{\$3.00}$.



b) Find the proportion of days when the price of the option is greater than $10.00.

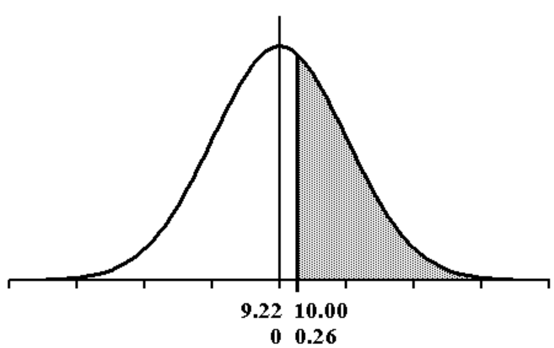
$$P(X > 10.00) = P\left(Z > \frac{10.00 - 9.22}{3.00}\right)$$

$= P(Z > 0.26)$

$= 1 - \Phi(0.26)$

$= 1 - 0.6026$

$= \mathbf{0.3974}$.



c) Following the famous "buy low, sell high" principle, the analyst recommends buying *Initech* stock option if the price falls into the lowest 14% of the price distribution, and selling if the price rises into the highest 9% of the distribution. Mr. Statman doesn't know much about history, doesn't know much about biology, doesn't know much about statistics, but he does want to be rich someday. Help Mr. Statman find the price below which he should buy *Initech* stock option and the price above which he should sell.

Need x = ? such that $P(X < x) = 0.14$.

① Find z such that $P(Z < z) = 0.14$.

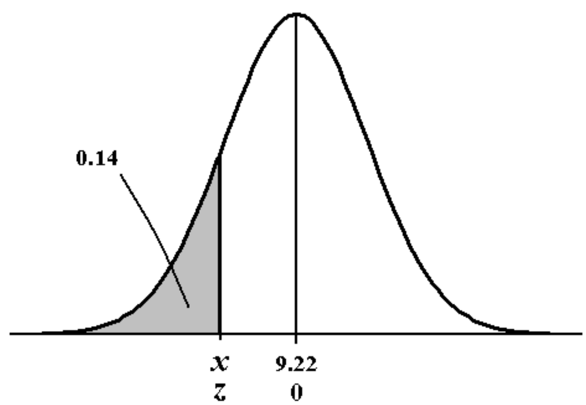
$z = \mathbf{-1.08}$.

② $x = \mu + \sigma \cdot z$.

$x = 9.22 + 3 \cdot (-1.08)$

$= \mathbf{\$5.98}$.

Buy if the price is below \$5.98.



Need x = ? such that $P(X > x) = 0.09$.

① Find z such that $P(Z > z) = 0.09$.

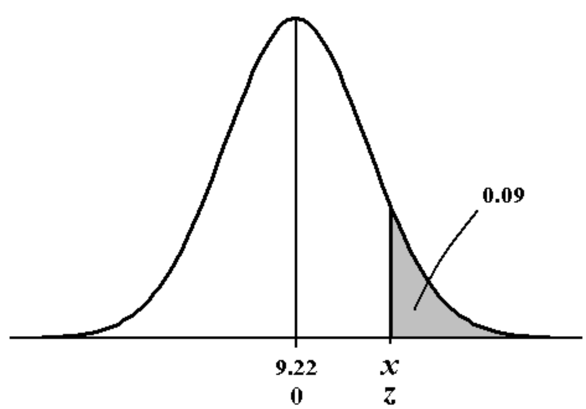
$z = \mathbf{1.34}$.

② $x = \mu + \sigma \cdot z$.

$x = 9.22 + 3 \cdot (1.34)$

$= \mathbf{\$13.24}$.

Sell if the price is above \$13.24.



2. Suppose X has a Normal distribution with mean $\mu = 3$ and variance $\sigma^2 = 16$.

$\mu = 3$ $\qquad \sigma^2 = 16 \quad \Rightarrow \quad \sigma = 4.$

a) Find $P(1 < X < 9)$.

$$P(1 < X < 9) = P\left(\frac{1-3}{4} < Z < \frac{9-3}{4}\right) = P(-0.50 < Z < 1.50)$$

$$= 0.9332 - 0.3085 = \mathbf{0.6247}.$$

b) Find $P(1 < X^2 < 9)$.

$$P(1 < X^2 < 9) = P(-3 < X < -1) + P(1 < X < 3)$$

$$= P\left(\frac{-3-3}{4} < Z < \frac{-1-3}{4}\right) + P\left(\frac{1-3}{4} < Z < \frac{3-3}{4}\right)$$

$$= P(-1.50 < Z < -1.00) + P(-0.50 < Z < 0.00)$$

$$= (0.1587 - 0.0668) + (0.5000 - 0.3085)$$

$$= 0.0919 + 0.1915 = \mathbf{0.2834}.$$

c) Find c such that $P(|X - 3| < c) = 0.9876$.

$P(3 - c < X < 3 + c) = 0.9876.$ $\qquad P(X < 3 - c) = \dfrac{1 - 0.9876}{2} = 0.0062.$

$P(Z < -2.50) = 0.0062.$ $\qquad \dfrac{3 - c - 3}{4} = \dfrac{-c}{4} = -2.50.$

$\Rightarrow \quad c = \mathbf{10}.$

3. **5.8-2** **10.5-2** **4.7-2**

If $E(X) = 17$ and $E(X^2) = 298$, use Chebyshev's inequality to determine

$$\mu = E(X) = 17, \qquad \sigma^2 = Var(X) = 298 - 17^2 = 9.$$

a) A lower bound for $P(10 < X < 24)$.

By Chebyshev's Inequality, for $\varepsilon > 0$, $\quad P(|X - \mu| < \varepsilon) \geq 1 - \dfrac{\sigma^2}{\varepsilon^2}$.

$$P(10 < X < 24) = P(|X - 17| < 7) \geq 1 - \frac{9}{7^2} = \mathbf{\frac{40}{49}} \approx 0.81632653.$$

b) An upper bound for $P(|X - 17| \geq 16)$.

By Chebyshev's Inequality, for $\varepsilon > 0$, $\quad P(|X - \mu| \geq \varepsilon) \leq \dfrac{\sigma^2}{\varepsilon^2}$.

$$P(|X - 17| \geq 16) \leq \frac{9}{16^2} = \mathbf{\frac{9}{256}} = 0.03515625.$$

4. Let X denote the place where a 4-inch chalk stick is broken into two pieces at random. Suppose X follows a Uniform distribution on interval $(0, 4)$.

a) Find the expected value of the length of the longer piece.

Hint 1: The length of the longer piece is a (piecewise-defined) function of X.

Hint 2: $\mathrm{E}(g(\mathrm{X})) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx.$

The lengths of the two pieces: x and $4 - x$.

The length of the longer piece $= g(x) = \begin{cases} 4 - x & 0 < x < 2 \\ x & 2 \le x < 4 \end{cases}$

$$\mathrm{E}(\text{longer}) = \mathrm{E}(g(\mathrm{X})) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx = \int_0^4 g(x) \cdot \frac{1}{4}\, dx$$

$$= \int_0^2 (4 - x) \cdot \frac{1}{4}\, dx + \int_2^4 x \cdot \frac{1}{4}\, dx$$

$$= \left(x - \frac{x^2}{8} \right) \Bigg|_0^2 + \left(\frac{x^2}{8} \right) \Bigg|_2^4 = \left(2 - \frac{1}{2} \right) + \left(2 - \frac{1}{2} \right) = \mathbf{3}.$$

b) Find the expected value of the length of the shorter piece.

$$\mathrm{E}(\text{shorter}) = 4 - \mathrm{E}(\text{longer}) = 4 - 3 = \mathbf{1}.$$

OR

$$\mathrm{E}(\text{shorter}) = \int_0^2 x \cdot \frac{1}{4}\, dx + \int_2^4 (4 - x) \cdot \frac{1}{4}\, dx = \ldots$$

5. Let T denote the time it takes for a computer to shut down. Suppose T follows an Exponential distribution with mean 0.80 minutes. A computer lab has 6 independent computers that must all be shut down at the end of the day.

a) What is the probability that at most 2 computers take longer than 1 minute to shut down on a given day?

$$P(T > 1) = e^{-1/0.80} = e^{-1.25} \approx 0.2865.$$

Let X denote the number of computers (out of 6) that take longer than 1 minute to shut down. Then X has a Binomial distribution, $n = 6$, $p = e^{-1.25} \approx 0.2865$.

$$P(X \le 2) = {}_6C_0 (e^{-1.25})^0 (1 - e^{-1.25})^6 + {}_6C_1 (e^{-1.25})^1 (1 - e^{-1.25})^5 + {}_6C_2 (e^{-1.25})^2 (1 - e^{-1.25})^4$$

$$\approx 0.1319 + 0.3179 + 0.3191 = \mathbf{0.7689}.$$

b) What is the probability that at least 2 computers take longer than 2 minutes to shut down on a given day?

$$P(T > 2) = e^{-2/0.80} = e^{-2.5} \approx 0.0821.$$

Let Y denote the number of computers (out of 6) that take longer than 2 minute to shut down. Then Y has a Binomial distribution, $n = 6$, $p = e^{-2.5} \approx 0.0821$.

$$P(Y \ge 2) = 1 - {}_6C_0 (e^{-2.5})^0 (1 - e^{-2.5})^6 - {}_6C_1 (e^{-2.5})^1 (1 - e^{-2.5})^5$$

$$\approx 1 - 0.59816 - 0.32094 = \mathbf{0.0809}.$$

6. a) Let $\theta > 0$. Suppose X has an Exponential distribution with mean θ.
Find $P(\mu - \sigma < X < \mu + \sigma)$.

Exponential $\Rightarrow$ $\mu = \theta$, $\sigma^2 = \theta^2$, $\sigma = \theta$.

$$P(\mu - \sigma < X < \mu + \sigma) = P(0 < X < 2\theta) = 1 - P(X \ge 2\theta)$$

$$= 1 - e^{-2\theta/\theta} = 1 - e^{-2} \approx \mathbf{0.864665}.$$

b) Let $\theta > 0$. Suppose X has a Uniform distribution on interval $(0, \theta)$.
Find $P(\mu - \sigma < X < \mu + \sigma)$.

Uniform$(0, \theta)$ $\Rightarrow$ $\mu = \dfrac{\theta}{2}$, $\sigma^2 = \dfrac{\theta^2}{12}$, $\sigma = \dfrac{\theta}{\sqrt{12}}$.

$$P(\mu - \sigma < X < \mu + \sigma) = P\left(\frac{\theta}{2} - \frac{\theta}{\sqrt{12}} < X < \frac{\theta}{2} + \frac{\theta}{\sqrt{12}}\right) = \frac{1}{\theta} \times \left(2 \times \frac{\theta}{\sqrt{12}}\right) = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}.$$

7. Consider a system consisting of three independent components. Suppose the lifetimes of these components follow an Exponential distribution with mean 200 hours, 250 hours, and 500 hours, respectively.

a) Suppose that the system works only if all three components are functional (e.g., series connection). Find the probability that the system still works after 300 hours. That is, find the probability that all three components still work after 300 hours.

"Hint": $P(X_1 > 300 \text{ AND } X_2 > 300 \text{ AND } X_3 > 300)$.

$$P(X_1 > 300 \cap X_2 > 300 \cap X_3 > 300) = P(X_1 > 300) \bullet P(X_2 > 300) \bullet P(X_3 > 300)$$

$$= e^{-300/200} \bullet e^{-300/250} \bullet e^{-300/500} = \boldsymbol{e^{-3.3}} \approx 0.0369.$$

b) Suppose that the system works only if at least one component is functional (e.g., parallel connection). Find the probability that the system still works after 300 hours. That is, find the probability that at least one component still work after 300 hours.

"Hint": $P(X_1 > 300 \text{ OR } X_2 > 300 \text{ OR } X_3 > 300)$.

$$P(X_1 > 300 \text{ OR } X_2 > 300 \text{ OR } X_3 > 300) = 1 - P(X_1 \le 300 \cap X_2 \le 300 \cap X_3 \le 300)$$

$$= 1 - P(X_1 \le 300) \bullet P(X_2 \le 300) \bullet P(X_3 \le 300)$$

$$= 1 - \left(1 - e^{-300/200}\right) \bullet \left(1 - e^{-300/250}\right) \bullet \left(1 - e^{-300/500}\right) \approx 0.755.$$

8 – 9. In Neverland, annual income (in \$) is distributed according to Gamma distribution with mean $\mu = \$40{,}000$ and standard deviation $\sigma = \$20{,}000$.

8. a) Find the parameters α and θ of Gamma distribution.

$$\mu = \alpha\,\theta = 40{,}000 \qquad \sigma^2 = \alpha\,\theta^2 = 20{,}000^2$$

$$\Rightarrow \qquad \alpha = \mathbf{4}, \qquad \theta = \mathbf{10{,}000}.$$

b) Find $P(X < \$26{,}000)$, $P(\$26{,}000 < X < \$50{,}000)$, $P(\$50{,}000 < X < \$85{,}000)$, and $P(X > \$85{,}000)$.

If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then

$$P(T_\alpha \le t) = P(X_t \ge \alpha) \quad \text{and} \quad P(T_\alpha > t) = P(X_t \le \alpha - 1),$$

where X_t has a Poisson(λt) distribution.

$$P(X > \$26{,}000) = P(\text{Poisson}(2.6) \le 3) = 0.736,$$

$$P(X > \$50{,}000) = P(\text{Poisson}(5.0) \le 3) = 0.265,$$

$$P(X > \$85{,}000) = P(\text{Poisson}(8.5) \le 3) = 0.030.$$

$$P(X < \$26{,}000) = 1 - 0.736 = \mathbf{0.264},$$

$$P(\$26{,}000 < X < \$50{,}000) = 0.736 - 0.265 = \mathbf{0.471},$$

$$P(\$50{,}000 < X < \$85{,}000) = 0.265 - 0.030 = \mathbf{0.235},$$

$$P(X > \$85{,}000) = \mathbf{0.030}.$$

9. 8. (continued)

Every year, the IRS audits 0.5% of the individuals with income below \$26,000, 1% of the individuals with income between \$26,000 and \$50,000, 3% of the individuals with income between \$50,000 and \$85,000, and 6.4% of the individuals with income above \$85,000. Suppose that the individuals to be audited at selected at random.

c) What proportion of Neverland's population is audited each year?

$$P(\text{Audit} \mid X < \$26{,}000) = 0.005,$$

$$P(\text{Audit} \mid \$26{,}000 < X < \$50{,}000) = 0.01,$$

$$P(\text{Audit} \mid \$50{,}000 < X < \$85{,}000) = 0.03,$$

$$P(\text{Audit} \mid X > \$85{,}000) = 0.064.$$

$$P(\text{Audit}) = 0.264 \times 0.005 + 0.471 \times 0.01 + 0.235 \times 0.03 + 0.030 \times 0.064$$

$$= 0.00132 + 0.00471 + 0.00705 + 0.00192 = \mathbf{0.015}.$$

1.5%

d) You have overheard Mr. Statman complain about being audited. What is the probability that Mr. Statman's income is below \$26,000? Between \$26,000 and \$50,000? Between \$50,000 and \$85,000? Above \$85,000?
["Hint": The answers should add up to 100%.]

$$P(X < \$26{,}000 \mid \text{Audit}) = \frac{0.00132}{0.015} = \mathbf{0.088},$$ **8.8%**

$$P(\$26{,}000 < X < \$50{,}000 \mid \text{Audit}) = \frac{0.00471}{0.015} = \mathbf{0.314},$$ **31.4%**

$$P(\$50{,}000 < X < \$85{,}000 \mid \text{Audit}) = \frac{0.00705}{0.015} = \mathbf{0.470},$$ **47.0%**

$$P(X > \$85{,}000 \mid \text{Audit}) = \frac{0.00192}{0.015} = \mathbf{0.128}.$$ **12.8%**

10. Alex purchased a laptop computer at *Joe's Discount Store*. He also purchased "Lucky 7" warranty plan that would replace the laptop at no cost if it needs 7 or more repairs in 3 years. Suppose the laptop requires repairs according to a Poisson process with an average rate of one repair per 6 month.

X_t = number of repairs in t years. Poisson(λt)

T_k = time of the kth repair. Gamma, $\alpha = k$.

one repair per 6 month $\Rightarrow$ $\lambda = 2$.

a) Find the probability that the laptop would not need to be replaced. That is, find the probability that the seventh time the laptop needs repair will be after 3 years, when the warranty expires.

$$P(T_7 > 3) = P(X_3 \le 6) = P(\text{Poisson}(6) \le 6) = \mathbf{0.606}.$$

OR

$$P(T_7 > 3) = \int_3^\infty \frac{2^7}{\Gamma(7)} t^{7-1} e^{-2t}\, dt = \int_3^\infty \frac{2^7}{6!} t^6 e^{-2t}\, dt = \ldots$$

b) Find the probability that the seventh time the laptop needs repair will be during the second year of warranty.

$$P(1 < T_7 < 2) = P(T_7 > 1) - P(T_7 > 2) = P(X_1 \le 6) - P(X_2 \le 6)$$

$$= P(\text{Poisson}(2) \le 6) - P(\text{Poisson}(4) \le 6) = 0.995 - 0.889 = \mathbf{0.106}.$$

OR

$$P(1 < T_7 < 2) = \int_1^2 \frac{2^7}{\Gamma(7)} t^{7-1} e^{-2t}\, dt = \int_1^2 \frac{2^7}{6!} t^6 e^{-2t}\, dt = \ldots$$

c) Find the probability that the seventh time the laptop needs repair will be during the last 6 month of the warranty period.

$$P(2.5 < T_7 < 3) = P(T_7 > 2.5) - P(T_7 > 3) = P(X_{2.5} \le 6) - P(X_3 \le 6)$$

$$= P(\text{Poisson}(5) \le 6) - P(\text{Poisson}(6) \le 6) = 0.762 - 0.606 = \mathbf{0.156}.$$

OR

$$P(2.5 < T_7 < 3) = \int_{2.5}^{3} \frac{2^7}{\Gamma(7)} t^{7-1} e^{-2t}\, dt = \int_{2.5}^{3} \frac{2^7}{6!} t^6 e^{-2t}\, dt = \ldots$$