STAT 400
UIUC

Practice Problems #9

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The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1. (Capture – Recapture) To estimate the populations size of unicorns in Neverland, first N_1 unicorns were captured and tagged. The captured unicorns were then released. One month later, n unicorns were captured. Let X denote the number of tagged unicorns among the ones in the second sample.

a) Construct an estimator for the population size N. Hint: Ask MoM.

b) Suppose $N_1 = 12$, $n = 10$, and $x = 3$. Obtain $\tilde{N}$, an estimate for the population size N.

c) Suppose $N = 33$, $N_1 = 12$, and $n = 10$. Find the probability that $\tilde{N}$ is within 10 of N. That is, find the probability $P(23 \le \tilde{N} \le 43)$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\lambda) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}}\, e^{-\lambda x^2}, \quad x > 0, \qquad \lambda > 0.$$

a) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

d) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find the maximum likelihood estimate of λ.

c) Obtain the method of moments estimator of λ, $\tilde{\lambda}$.

d) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find a method of moments estimate of λ.

3. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f_X(x) = f_X(x;\theta) = \left(\theta^2 + \theta\right) x^{\theta-1}(1-x), \qquad 0 < x < 1, \qquad \theta > 0.$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

b) Is $\tilde{\theta}$ an unbiased estimator of θ? *Justify your answer.*

c) Suppose $n = 6$, and $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.6$, $x_4 = 0.65$, $x_5 = 0.75$, $x_6 = 0.8$. Find a method of moments estimate of θ.

4. Let $\theta > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from a Uniform distribution on interval $(0, \theta)$.

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

b) Is $\tilde{\theta}$ an unbiased estimator of θ? *Justify your answer.*

c) Find $\operatorname{Var}(\tilde{\theta})$.

d) Find $\operatorname{MSE}(\tilde{\theta})$.

5. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{3 + \theta x}{12 + 8\theta}, \qquad 0 < x < 4, \qquad \theta > -\frac{3}{4}.$$

a) Find the method of moments estimator of θ, $\tilde{\theta}$.

b) Suppose $n = 5$, and $x_1 = 1.2$, $x_2 = 1.8$, $x_3 = 2.6$, $x_4 = 3.1$, $x_5 = 3.8$. Find the method of moments estimate of θ.

c) Suppose $n = 4$, and $x_1 = 1.3$, $x_2 = 2.2$, $x_3 = 3.1$, $x_4 = 3.8$. Find the method of moments estimate of θ.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from Binomial$(1, p)$ (i.e., n Bernoulli trials). Thus

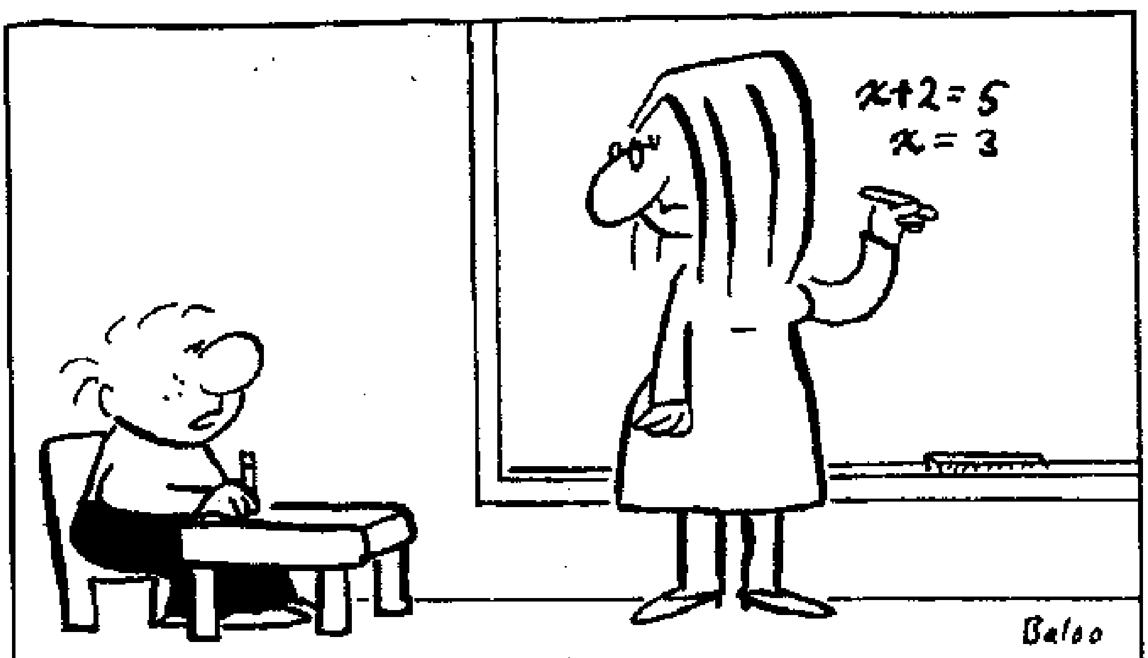
$$Y = \sum_{i=1}^{n} X_i \quad \text{is Binomial}(n, p).$$

a) Show that $\overline{X} = \dfrac{Y}{n}$ is an unbiased estimator of p.

b) Show that $\operatorname{Var}(\overline{X}) = \dfrac{p(1-p)}{n}$.

c) Show that $E\left[\dfrac{\overline{X}(1-\overline{X})}{n}\right] = (n-1)\left[\dfrac{p(1-p)}{n^2}\right]$.

d) Find the value of c so that $c\,\overline{X}(1-\overline{X})$ is an unbiased estimator of $\operatorname{Var}(\overline{X}) = \dfrac{p(1-p)}{n}$.



"Just a darn minute! Yesterday you said X equals two!"

7 – 8. Let $\lambda > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with the probability density function

$$f(x;\lambda) = 2\lambda^2 x^3 e^{-\lambda x^2}, \qquad x > 0.$$

7. a) Find $E(X^k)$, $k > -4$.

Hint 1: Consider $u = \lambda x^2$ or $u = x^2$.

Hint 2: $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du$, $a > 0$.

Hint 3: $\Gamma(a) = (a-1)\,\Gamma(a-1)$.

Hint 4: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

b) Obtain a method of moments estimator of λ, $\tilde{\lambda}$.

c) Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.
Find a method of moments estimate of λ.

8. d) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

e) Suppose $n = 5$, and $x_1 = 0.6$, $x_2 = 1.1$, $x_3 = 2.7$, $x_4 = 3.3$, $x_5 = 4.5$.
Find the maximum likelihood estimate of λ.

f) Suppose $n = 5$, and $x_1 = 0.5$, $x_2 = 1.2$, $x_3 = 0.4$, $x_4 = 0.8$, $x_5 = 0.1$.
Find the maximum likelihood estimate of λ.

9. A random sample of size $n = 16$ from $N(\mu, \sigma^2 = 64)$ yielded $\bar{x} = 85$.
Construct the following confidence intervals for μ:

a) 95%. b) 90%. c) 80%.

10. What is the minimum sample size required for estimating μ for $N(\mu, \sigma^2 = 64)$ to within ± 3 with confidence level

a) 95%. b) 90%. c) 80%.