STAT 400
UIUC

Practice Problems #10

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SOLUTIONS

The following are a number of practice problems that may be *helpful* for completing the homework, and will likely be **very useful** for studying for exams.

1 – 3. A store sells "16-ounce" boxes of *Captain Crisp* cereal. A random sample of 9 boxes was taken and weighed. The results were the following (in ounces):

15.5	16.2	16.1	15.8	15.6	16.0	15.8	15.9	16.2

Assume the weight of cereal in a box is normally distributed.

1. a) Compute the sample mean $\overline{x}$ and the sample standard deviation s.
Do NOT use a computer. You may only use +, –, ×, ÷, and $\sqrt{\ }$ on a calculator.
Show ALL work.

$$\overline{x} = \frac{\sum x_i}{n} = \frac{15.5+16.2+16.1+15.8+15.6+16.0+15.8+15.9+16.2}{9} = \frac{143.1}{9} = \mathbf{15.9}.$$

x	x^2
15.5	240.25
16.2	262.44
16.1	259.21
15.8	249.64
15.6	243.36
16.0	256.00
15.8	249.64
15.9	252.81
16.2	262.44
	2275.79

OR

x	$x-\overline{x}$	$\left(x-\overline{x}\right)^2$
15.5	– 0.4	0.16
16.2	0.3	0.09
16.1	0.2	0.04
15.8	– 0.1	0.01
15.6	– 0.3	0.09
16.0	0.1	0.01
15.8	– 0.1	0.01
15.9	0.0	0.00
16.2	0.3	0.09
	0	0.50

$$s^2 = \frac{\sum x_i^2 - \left(\sum x_i\right)^2 \big/ n}{n-1} = \frac{2275.79 - \left(143.1\right)^2 \big/ 9}{8} = \frac{0.5}{8} = 0.0625.$$

OR

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = \frac{0.5}{8} = 0.0625.$$

$$s = \sqrt{s^2} = \sqrt{0.0625} = \mathbf{0.25}.$$

2. b) Construct a 95% confidence interval for the overall average weight of boxes of *Captain Crisp* cereal.

σ is unknown. $n = 9$ - small. The confidence interval : $\overline{X} \pm t_{\alpha/2} \dfrac{s}{\sqrt{n}}$.

number of degrees of freedom = $n - 1 = 9 - 1 = 8$.

$\alpha = 0.05$ $\alpha/2 = 0.025$. $t_{\alpha/2} = t_{0.025} = 2.306$.

$15.9 \pm 2.306 \cdot \dfrac{0.25}{\sqrt{9}}$ $\mathbf{15.9 \pm 0.192}$ **(15.708 ; 16.092)**

c) Construct a 95% confidence upper bound for the overall average weight of boxes of *Captain Crisp* cereal.

$\left(0,\ \overline{X} + t_{\alpha} \dfrac{s}{\sqrt{n}}\right)$ 8 degrees of freedom $t_{\alpha} = t_{0.05} = 1.860$.

$\left(0,\ 15.9 + 1.860 \cdot \dfrac{0.25}{\sqrt{9}}\right)$ **(0 ; 16.055)**

d) Construct a 90% confidence lower bound for the overall average weight of boxes of *Captain Crisp* cereal.

$\left(\overline{X} - t_{\alpha} \dfrac{s}{\sqrt{n}},\ \infty\right)$ 8 degrees of freedom $t_{\alpha} = t_{0.10} = 1.397$.

$\left(15.9 - 1.397 \cdot \dfrac{0.25}{\sqrt{9}},\ \infty\right)$ **(15.7836 ; ∞)**

3. e) Construct a 90% confidence interval for the overall standard deviation of the weights of boxes of *Captain Crisp* cereal.

Confidence Interval for σ^2 : $$\left(\frac{(n-1)\cdot s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha/2}} \right).$$

$\alpha = 0.10.$ $\qquad \alpha/2 = 0.05.$ $\qquad 1 - \alpha/2 = 0.95.$

number of degrees of freedom = $n - 1 = 9 - 1 = 8.$

$\chi^2_{\alpha/2} = 15.51.$ $\qquad \chi^2_{1-\alpha/2} = 2.733.$

$$\left(\frac{(9-1)\cdot 0.0625}{15.51}, \frac{(9-1)\cdot 0.0625}{2.733} \right) \qquad (\ 0.032\ ;\ 0.183\ )$$

Confidence Interval for σ : $\left(\sqrt{0.032}, \sqrt{0.183} \right)$ = **(0.179 ; 0.428)**

f) Construct the 90% minimum length confidence interval for the overall standard deviation of the weights of boxes of *Captain Crisp* cereal.

g) Construct a 99% confidence lower bound for the overall standard deviation of the weights of boxes of *Captain Crisp* cereal.

$$\left(\sqrt{\frac{(n-1)\cdot s^2}{\chi^2_{\alpha}}}, \infty \right)$$ 8 degrees of freedom $\qquad \chi^2_{0.01} = 20.09.$

$$\left(\sqrt{\frac{(9-1)\cdot 0.0625}{20.09}}, \infty \right) \qquad \mathbf{(\ 0.15776\ ;\ \infty\ )}$$

For fun:

99% conf. upper bound for σ: $$\left(0, \sqrt{\frac{(n-1)\cdot s^2}{\chi^2_{1-\alpha}}} \right) = \left(0, \sqrt{\frac{(9-1)\cdot 0.0625}{1.646}} \right) = (\ 0, 0.551\ )$$

4. a) What is the minimum sample size required to estimate the overall mean weight of boxes of *Captain Crisp* cereal to within 0.01 ounces with 95% confidence, if the overall (population) standard deviation of the weights is 0.20 ounces?

$\sigma = 0.20.$ $\varepsilon = 0.01.$ $\alpha = 0.05.$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = 1.960.$

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{1.960 \cdot 0.20}{0.01} \right)^2 = 1536.64.$$

Round up. $n =$ **1537**.

b) What is the minimum sample size required to estimate the overall proportion of boxes that have less than 16 ounces of cereal to within 2% with 95% confidence, if no guess as to the value of this proportion is available?

Use $p = 0.50.$ $\varepsilon = 0.02.$ $\alpha = 0.05.$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = 1.960.$

$$n = \left(\frac{z_{\alpha/2}}{\varepsilon} \right)^2 \cdot p \cdot (1 - p) = \left(\frac{1.960}{0.02} \right)^2 \cdot 0.50 \cdot 0.50 = \mathbf{2401}.$$

c) What is the minimum sample size required to estimate the overall proportion of boxes that have less than 16 ounces of cereal to within 2% with 95% confidence, if it is known that this proportion is at most 0.30?

Use $p = 0.30$ (closest to 0.50 possible value). $\varepsilon = 0.02.$

$\alpha = 0.05.$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = 1.960.$

$$n = \left(\frac{z_{\alpha/2}}{\varepsilon} \right)^2 \cdot p \cdot (1 - p) = \left(\frac{1.960}{0.02} \right)^2 \cdot 0.30 \cdot 0.70 = 2016.84.$$

Round up. $n =$ **2017**.

5 – 6. A random sample of 144 patients, suffering from a particular disease are given a new medicine. 108 of the patients report an improvement in their condition.

5. a) Construct a 90% (two-sided) confidence interval for the overall improvement rate of this medicine.

X = 108. n = 144. $\hat{p} = \frac{X}{n} = \frac{108}{144} = 0.75.$

The confidence interval : $\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$.

90% confidence level $\alpha = 0.10$ $\alpha/2 = 0.05.$ $z_{\alpha/2} = 1.645.$

$0.75 \pm 1.645 \cdot \sqrt{\frac{(0.75)(0.25)}{144}}$ $\mathbf{0.75 \pm 0.05936}$ **(0.69 , 0.81)**

b) Find the minimum sample size required if we want to estimate the improvement rate of this medicine to within 4% with 90% confidence if it is known that the improvement rate is between 70% and 90%.

Use $p^* = 0.70$ (closest to 0.50 possible value). $\varepsilon = 0.04.$

$$n = \left(\frac{z_{\alpha/2}}{\varepsilon} \right)^2 \cdot p^* \cdot (1 - p^*) = \left(\frac{1.645}{0.04} \right)^2 \cdot 0.70 \cdot 0.30 = 355.1658.$$

Round **up**. n = **356**.

c) Find the minimum sample size required if we want to estimate the improvement rate of this medicine to within 4% with 90% confidence if we do not make any assumptions about the improvement rate.

Use $p^* = 0.50.$ $\varepsilon = 0.04.$

$$n = \left(\frac{z_{\alpha/2}}{\varepsilon} \right)^2 \cdot p^* \cdot (1 - p^*) = \left(\frac{1.645}{0.04} \right)^2 \cdot 0.50 \cdot 0.50 = 422.8164.$$

Round **up**. n = **423**.

6. d) Construct a 90% confidence lower bound for the overall improvement rate.

$$\left(\hat{p} - z_\alpha \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}} \,,\ 1 \right) \qquad \left(0.75 - 1.282 \cdot \sqrt{\frac{(0.75)(0.25)}{144}} \,,\ 1 \right)$$

(0.704 , 1)

e) Construct a 95% confidence upper bound for the overall improvement rate.

$$\left(0 ,\ \hat{p} + z_\alpha \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}} \right) \qquad \left(0 ,\ 0.75 + 1.645 \cdot \sqrt{\frac{(0.75)(0.25)}{144}} \right)$$

(0 , 0.81)

f) The company that manufactures this medicine claims an 80% improvement rate. Based on your answer to part (e), do you find this claim believable? Explain your answer.

The sample proportion does not have to be equal to the overall population proportion. It is likely to be a bit smaller or a bit larger than the overall population proportion. It could be a lot smaller or a lot larger than the overall population proportion, but this is unlikely to happen just by chance. The fact that the sample proportion, 75%, is smaller than the claimed 80% alone does not yet allow us to draw conclusions about the credibility of the manufacturer's claim. To do this, we should decide whether 75% is a little bit smaller than 80% or a lot smaller than 80%.

Based on the information in our sample and part (e), values as high as 81% are reasonable candidates for the value of the (unknown) overall improvement rate of this medicine. Therefore, the manufacturer's claim that this rate is 80% is believable.

7. The United States Secret Service (USSS) hires you to compute a 95% confidence interval for the overall average maximum weight (in pounds) its agents can bench press in a 5-repetition set. Stressing the top secret nature of the data, the USSS does not make the data available to you, it does not even report the values of the sample mean and the sample standard deviation. The only information you have is that an 80% confidence interval based on a sample of 14 agents is $(250, 277)$. Assume the population is approximately normally distributed. Compute a 95% confidence interval for the overall average maximum weight USSS agents can bench press in a 5-repetition set.

σ is unknown. The confidence interval : $\overline{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$.

$\alpha = 0.20$. $\alpha/2 = 0.10$. number of degrees of freedom = $n - 1 = 14 - 1 =$ **13**.

$t_{0.10} =$ **1.350**. $\overline{X} - 1.350 \cdot \frac{s}{\sqrt{14}} = 250$. $\overline{X} + 1.350 \cdot \frac{s}{\sqrt{14}} = 277$.

$\overline{X}$ is in the middle of the confidence interval. $\overline{X} = \frac{250 + 277}{2} = \mathbf{263.5}$.

$263.5 - 1.350 \cdot \frac{s}{\sqrt{14}} = 250$. $s = \mathbf{37.42}$.

$\overline{X} = 263.5$, $s = 37.42$, $n = 14$.

σ is unknown. The confidence interval : $\overline{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$.

$\alpha = 0.05$. $\alpha/2 = 0.025$. number of degrees of freedom = $n - 1 = 14 - 1 =$ **13**.

$t_{0.025} =$ **2.160**.

$263.5 \pm 2.160 \cdot \frac{37.42}{\sqrt{14}}$ **263.5 ± 21.6** **(241.9 ; 285.1)**

8 – 10. A random sample of size $n = 289$ from a $N(\mu, \sigma^2)$ distribution gives the sample mean $\overline{x} = 789$ and the sample standard deviation $s = 34$.

8. If the number of degrees of freedom (d.f.) is large, the values of $t_{\alpha/2}$ can be approximated by $z_{\alpha/2}$.

a) Compare $z_{0.05} = 1.645$ to the value of $t_{0.05}(n-1 \text{ d.f.})$.

"Hint": EXCEL =TINV(*two*-tail probability , degrees of freedom).

That is, =TINV(α , $n - 1$).

OR R `> qt(` probability to the *left* `,` degrees of freedom `)`.

That is, `> qt(` $1 - \alpha/2$ `,` $n - 1$ `)`.

$t_{0.05}(288)$ = **1.650162**. (close to $z_{0.05} = 1.645$)

b) Construct a 90% (two-sided) confidence interval for μ.

$\overline{X} \pm t_{\alpha/2} \dfrac{s}{\sqrt{n}}$ $789 \pm 1.65 \dfrac{34}{\sqrt{289}}$ **789 ± 3.30** **(785.70, 792.30)**

OR

$\overline{X} \pm z_{\alpha/2} \dfrac{s}{\sqrt{n}}$ $789 \pm 1.645 \dfrac{34}{\sqrt{289}}$ **789 ± 3.29** **(785.71, 792.29)**

c) Compare $z_{0.025} = 1.96$ to the value of $t_{0.025}(n-1 \text{ d.f.})$.

$t_{0.025}(288)$ = **1.968235**. (close to $z_{0.025} = 1.96$)

d) Construct a 95% (two-sided) confidence interval for μ.

$\overline{X} \pm t_{\alpha/2} \dfrac{s}{\sqrt{n}}$ $789 \pm 1.968 \dfrac{34}{\sqrt{289}}$ **789 ± 3.936** **(785.064, 792.936)**

OR

$\overline{X} \pm z_{\alpha/2} \dfrac{s}{\sqrt{n}}$ $789 \pm 1.96 \dfrac{34}{\sqrt{289}}$ **789 ± 3.92** **(785.08, 792.92)**

9. If the number of degrees of freedom (d.f.) is large, the values of chi-squared distribution can be approximated by the values of a Normal distribution with mean $\mu = \text{d.f.}$ and variance $\sigma^2 = 2 \times \text{d.f.}$

a) Find the two values of a Normal distribution $N(\mu = \text{d.f.}, \sigma^2 = 2 \times \text{d.f.})$ distribution with area 0.05 to the left and to the right, respectively. Here, $\text{d.f.} = n - 1$.

$\mu = 288$ $\sigma^2 = 2 \times 288 = 576$ $\sigma = 24$

$288 - 1.645 \times 24$ $288 + 1.645 \times 24$

248.52 **327.48**

b) Compare the answers from part (a) with the values of $\chi^2_{0.95}(n - 1 \text{ d.f.})$ and $\chi^2_{0.05}(n - 1 \text{ d.f.})$.

"Hint": EXCEL =CHIINV(probability to the *right* , degrees of freedom).

OR R `> qchisq(` probability to the *left* `,` degrees of freedom `)`.

$\chi^2_{0.95}(288 \text{ d.f.}) =$ **249.6928**. $\chi^2_{0.05}(288 \text{ d.f.}) =$ **328.5804**.

c) Construct a 90% (two-sided) confidence interval for σ.

$$\left(\sqrt{\frac{(289-1)\cdot 34^2}{328.58}}, \sqrt{\frac{(289-1)\cdot 34^2}{249.69}} \right)$$ **(31.8 ; 36.5)**

OR

$$\left(\sqrt{\frac{(289-1)\cdot 34^2}{327.48}}, \sqrt{\frac{(289-1)\cdot 34^2}{248.52}} \right)$$ **(31.9 ; 36.6)**

10. 9. (continued)

d) Find the probability $P(X > 312)$ using …

i) … $N(\mu = 288, \sigma^2 = 2 \times 288)$ distribution;

$N(\mu = 288, \sigma^2 = 2 \times 288) = N(\mu = 288, \sigma^2 = 24^2)$:

$$P(X > 312) = P\left(Z > \frac{312 - 288}{24}\right) = P(Z > 1.00) = \mathbf{0.1587}.$$

ii) … $\chi^2(288 \text{ d.f.})$ distribution.

"Hint": EXCEL =CHIDIST(x , degrees of freedom) gives area to the *right* of x.

OR R `> pchisq(` x , degrees of freedom `)` gives area to the *left* of x.

=CHIDIST(312,288) **0.158391**.

OR

```
> 1 - pchisq(312,df = 288)
[1] 0.1583908
```

e) Find the probability $P(270 < X < 300)$ using …

i) … $N(\mu = 288, \sigma^2 = 2 \times 288)$ distribution;

$$P(270 < X < 300) = P\left(\frac{270 - 288}{24} < Z < \frac{300 - 288}{24}\right)$$

$$= P(-0.75 < Z < 0.50) = 0.6915 - 0.2266 = \mathbf{0.4649}.$$

ii) … $\chi^2(288 \text{ d.f.})$ distribution.

$$P(270 < X < 300) = P(X > 270) - P(X > 300)$$

=CHIDIST(270,288)-CHIDIST(300,288) **0.468615**.

OR

```
> pchisq(300,df = 288) - pchisq(270,df = 288)
[1] 0.4686153
```