Discrete Random Variables, II

Named discrete distributions: The Big Three

The following is a list of essential formulas for the three most important discrete distributions: binomial, geometric, and Poisson. You are expected to know these formulas in exams, so you should memorize them. For p.m.f.'s be sure to also memorize the range (i.e., the set of values x at which $f(x)$ is defined), along with the formula for $f(x)$,

1. **Binomial distribution** $b(n,p)$**:**
   - **Parameters:** n (positive integer), p ($0 \le p \le 1$)
   - **P.m.f.:** $f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad (x = 0, 1, 2, \dots, n)$
     (see back of page for definitions and properties of the binomial coefficients $\binom{n}{k}$)
   - **Expectation and variance:** $\mu = np$, $\sigma^2 = np(1-p)$
   - **Arises as:** Distribution of number of successes in success/failure trials ("Bernoulli trials")
2. **Geometric distribution:**
   - **Parameter:** p ($0 < p < 1$)
   - **P.m.f.:** $f(x) = (1-p)^{x-1} p$ ($x = 1, 2, \dots$)
   - **Expectation and variance:** $\mu = 1/p$, $\sigma^2 = (1-p)/p^2$
   - **Geometric series formula:** $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ ($|r| < 1$)
   - **Arises as:** Distribution of trial at which the first success occurs in success/failure trial sequence
3. **Poisson distribution:**
   - **Parameter:** $\lambda > 0$
   - **P.m.f.:** $f(x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad (x = 0, 1, 2, \dots)$
   - **Expectation and variance:** $\mu = \lambda$, $\sigma^2 = \lambda$
   - **Exponential series formula:** $\displaystyle\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{\lambda}$
   - **Arises as:** Distribution of number of occurrences of **rare** events, such as accidents, insurance claims, etc.

Other discrete distributions

The following distributions are listed in the inside cover of Hogg/Tanis, but you need not memorize the various formulas associated with these distributions. These distributions are far less important and common than the above three, and you won't need them for any hw/quiz/exam problems.

1. **Hypergeometric distribution:** $f(x) = \dfrac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}, \quad x = 0, 1, \dots, N_1,\ n - x \le N_2$
2. **Negative binomial distribution:** $f(x) = \binom{x-1}{r-1}(1-p)^{x-r} p^r, \quad x = r, r+1, \dots$

Binomial coefficients

- **Definition:** For $n = 1, 2, \ldots$ and $k = 0, 1, \ldots, n$, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.
  (Note that, by definition, $0! = 1$.)
- **Alternate notations:** ${}_nC_k$ or $C(n, k)$
- **Alternate definition:** $\binom{n}{k} = \dfrac{n(n-1)\ldots(n-k+1)}{k!}$.
  (This version is convenient for hand-calculating binomial coefficients.)
- **Symmetry property:** $\binom{n}{k} = \binom{n}{n-k}$
- **Special cases:** $\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n$
- **Binomial Theorem:** $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- **Binomial Theorem, special case:** $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$
- **Combinatorial Interpretations:** $\binom{n}{k}$ represents
  1. the number of ways to select k objects out of n given objects (in the sense of unordered samples without replacement);
  2. the number of k-element subsets of an n-element set;
  3. the number of n-letter HT sequences with exactly k H's and $n-k$ T's.
- **Binomial distribution:** Given a positive integer n and a number p with $0 < p < 1$, the binomial distribution $b(n, p)$ is the distribution with density (p.m.f.) $f(x) = \binom{n}{x} p^x (1-p)^{n-x}$, for $x = 0, 1, \ldots, n$.