General Probability, II: Independence and conditional probability

Definitions and properties

1. **Independence:** A and B are called independent if they satisfy the *product formula*
$$P(A \cap B) = P(A)P(B).$$

2. **Conditional probability:** The *conditional probability of A given B* is denoted by $P(A|B)$ and defined by the formula
$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$
provided $P(B) > 0$. (If $P(B) = 0$, the conditional probability is not defined.)

3. **Independence of complements:** If A and B are independent, then so are A and B', A' and B, and A' and B'.

4. **Connection between independence and conditional probability:** If the conditional probability $P(A|B)$ is equal to the ordinary ("unconditional") probability $P(A)$, then A and B are independent. Conversely, if A and B are independent, then $P(A|B) = P(A)$ (assuming $P(B) > 0$).

5. **Complement rule for conditional probabilities:** $P(A'|B) = 1 - P(A|B)$. That is, with respect to the *first* argument, A, the conditional probability $P(A|B)$ satisfies the ordinary complement rule.

6. **Multiplication rule:** $P(A \cap B) = P(A|B)P(B)$

Some special cases

- If $P(A) = 0$ or $P(B) = 0$ then A and B are independent. The same holds when $P(A) = 1$ or $P(B) = 1$.
- If $B = A$ or $B = A'$, A and B are not independent except in the above trivial case when $P(A)$ or $P(B)$ is 0 or 1. In other words, an event A which has probability strictly between 0 and 1 is not independent of itself or of its complement.

Notes and hints

- **Formal versus intuitive notion of independence:** When working problems, always use the above formal mathematical definitions of independence and conditional probabilities. While these definitions are motivated by our intuitive notion of these concepts and *most of the time* consistent with what our intuition would predict, intuition, aside from being non-precise, does fail us some time and lead to wrong conclusions as illustrated, for example, by the various paradoxes in probability.

- **Independence is not the same as disjointness:** If A and B are disjoint (corresponding to mutually exclusive events), then the intersection $A \cap B$ is the empty set, so $P(A \cap B) = P(\emptyset) = 0$, so independence can only hold in the trivial case when one of the events has probability 0. While, at first glance, this might seem counterintuitive, it is, in fact, consistent with the interpretation of disjointness as meaning that A and B are mutually exclusive, that is, if A occurs, then B cannot occur, and vice versa.

- **Independence and conditional probabilities in Venn diagrams:** In contrast to other properties such as disjointness, independence can **not** be spotted in Venn diagrams. On the other hand, conditional probabilities have a natural interpretation in Venn diagrams: The conditional probability given B is the probability you get if the underlying sample space S is "shrunk" to the set B (i.e., everything outside B is deleted), and then rescaled so as to have again unit area.

- **Verbal descriptions of conditional probabilities:** Whether a probability in a word problem represents a conditional or ordinary (unconditional) probability is not always obvious, and you have to read the problem carefully to see which interpretation is the correct one. Typically, conditional probabilities are indicated by words like "given", "if", or "among" (e.g., in the context of subpopulations), though there are no hard rules, and it may depend on what the underlying universe (sample space) is, which is usually not explicitly stated, but should be clear from the context of the entire problem.

- **Don't make assumptions about independence:** If a problem does not explicitly state that two events are independent, they are probably not, and not you should not make any assumptions about independence.

- $P(A|B)$ **is not the same as** $P(B|A)$**:** In contrast to set-theoretic operations like union or intersection, in conditional probabilities the order of the sets matters.

- $P(A|B')$ **is not the same as** $1 - P(A|B)$**:** The complement formula only holds with respect to the first argument. There is no corresponding formula for $P(A|B')$.

- **Independence of three or more events:** Though rarely needed, the definition of independence of two events can be extended to three events as follows: A, B, C are called **mutually independent** if the product formula holds for (i) the intersection of all three events (i.e., $P(A \cap B \cap C) = P(A)P(B)P(C)$) and (ii) for any combination of two of these three events (i.e., $P(A \cap B) = P(A)P(B)$ and similarly for $P(A \cap C)$, $P(B \cap C)$). More generally, n events $A_1, \ldots, A_n$ are called independent if the product formula holds for any subcollection of these events.