

## Joint Distributions, Continuous Case

In the following,  $X$  and  $Y$  are **continuous** random variables. Most of the concepts and formulas below are analogous to those for the discrete case, with integrals replacing sums. The principal difference between continuous lies in the definition of the p.d.f./p.m.f.  $f(x, y)$ : The formula  $f(x, y) = P(X = x, Y = y)$  is no longer valid, and there is no simple and direct way to obtain  $f(x, y)$  from  $X$  and  $Y$ .

### 1. Joint continuous distributions:

- **Joint density (joint p.d.f.):** A function  $f(x, y)$  satisfying (i)  $f(x, y) \geq 0$ , (ii)  $\iint f(x, y) dx dy = 1$ . Usually,  $f(x, y)$  will be given by an explicit formula, along with a **range** (a region in the  $xy$ -plane) on which this formula holds. *In the general formulas below, if a range of integration is not explicitly given, the integrals are to be taken over the range in which the density function is defined.*
  - **Uniform joint distribution:** An important special type of joint density is one that is constant over a given range (a region in the  $xy$ -plane), and 0 outside outside this range, the constant being the reciprocal of the area of of the range. This is analogous to the concept of an ordinary (one-variable) uniform density  $f(x)$  over an interval  $I$ , which is constant (and equal to the reciprocal of the length of  $I$ ) inside the interval, and 0 outside it.
2. **Marginal distributions:** The ordinary distributions of  $X$  and  $Y$ , when considered separately. The corresponding (one-variable) densities are denoted by  $f_X$  (or  $f_1$ ) and  $f_Y$  (or  $f_2$ ), and obtained by integrating the joint density  $f(x, y)$  over the “other” variable:

$$f_X(x) = \int f(x, y) dy, \quad f_Y(y) = \int f(x, y) dx.$$

### 3. Computations with joint distributions:

- **Probabilities:**

Given a region  $R$  in the  $xy$ -plane the probability that  $(X, Y)$  falls into this region is given by the double integral of  $f(x, y)$  over this region. For example,  $P(X + Y \leq 1)$  is given by an integral of the form  $\iint_R f(x, y) dx dy$ , where  $R$  consists of the part of the range of  $f$  in which  $x + y \leq 1$ .

- **Expectation of a function of  $X$  and  $Y$  (e.g.,  $u(x, y) = xy$ ):**  
 $E(u(X, Y)) = \iint u(x, y) f(x, y) dx dy$

### 4. Covariance and correlation:

The formulas and definitions are the same as in the discrete case.

- **Definitions:**  $\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = E((X - \mu_X)(Y - \mu_Y))$  (Covariance of  $X$  and  $Y$ ),  $\rho = \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$  (Correlation of  $X$  and  $Y$ )
- **Properties:**  $|\text{Cov}(X, Y)| \leq \sigma_X \sigma_Y$ ,  $-1 \leq \rho(X, Y) \leq 1$
- **Relation to variance:**  $\text{Var}(X) = \text{Cov}(X, X)$
- **Variance of a sum:**  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

### 5. Independence of random variables: Same as in the discrete case:

- **Definition:**  $X$  and  $Y$  are called independent if the joint p.d.f. is the product of the individual p.d.f.'s: i.e., if  $f(x, y) = f_X(x)f_Y(y)$  for all  $x, y$ .

- **Properties of independent random variables:**

If  $X$  and  $Y$  are independent, then:

- **The expectation of the product of  $X$  and  $Y$  is the product of the individual expectations:**  $E(XY) = E(X)E(Y)$ . More generally, this product formula holds for any expectation of a function  $X$  times a function of  $Y$ . For example,  $E(X^2Y^3) = E(X^2)E(Y^3)$ .
- **The product formula holds for probabilities of the form  $P(\text{some condition on } X, \text{ some condition on } Y)$  (where the comma denotes “and”):** For example,  $P(X \leq 2, Y \leq 3) = P(X \leq 2)P(Y \leq 3)$ .
- **The covariance and correlation of  $X$  and  $Y$  are 0:**  $\text{Cov}(X, Y) = 0$ ,  $\rho(X, Y) = 0$ .
- **The variance of the sum of  $X$  and  $Y$  is the sum of the individual variances:**  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
- **The moment-generating function of the sum of  $X$  and  $Y$  is the product of the individual moment-generating functions:**  $M_{X+Y}(t) = M_X(t)M_Y(t)$ .

6. **Conditional distributions:** Same as in the discrete case, with integrals in place of sums:

- **Definitions:**

- **conditional density of  $X$  given that  $Y = y$ :**  

$$g(x|y) = \frac{f(x,y)}{f_Y(y)}$$
- **conditional density of  $Y$  given that  $X = x$ :**  

$$h(y|x) = \frac{f(x,y)}{f_X(x)}$$

- **Conditional expectations and variance:** Conditional expectations, variances, etc., are defined and computed as usual, but with conditional distributions in place of ordinary distributions. For example:

- $E(X|Y = 1) = E(X|Y = 1) = \int xg(x|1)dx$
- $E(X^2|Y = 1) = E(X^2|Y = 1) = \int x^2g(x|1)dx$