Variance, covariance, and moment-generating functions

Definitions and basic properties

- **Basic definitions:**
  - **Variance:** $\operatorname{Var}(X) = \mathrm{E}(X^2) - E(X)^2$
  - **Covariance:** $\operatorname{Cov}(X, Y) = E(XY) - E(X)E(Y)$
  - **Correlation:** $\rho = \rho(X, Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}$
  - **Moment-generating function (mgf):** $M(t) = M_X(t) = \mathrm{E}(e^{tX})$
- **General properties:**
  - $E(c) = c$, $E(cX) = cE(X)$
  - $\operatorname{Var}(c) = 0$, $\operatorname{Var}(cX) = c^2 \operatorname{Var}(X)$, $\operatorname{Var}(X + c) = \operatorname{Var}(X)$
  - $M(0) = 1$, $M'(0) = \mathrm{E}(X)$, $M''(0) = \mathrm{E}(X^2)$, $M'''(0) = \mathrm{E}(X^3)$, etc.
  - $E(X + Y) = E(X) + E(Y)$
  - $\operatorname{Var}(X + Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X, Y)$.
- **Additional properties holding for independent r.v.'s X and Y:**
  - $E(XY) = E(X)E(Y)$
  - $\operatorname{Cov}(X, Y) = 0$
  - $\operatorname{Var}(X + Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$
  - $M_{X+Y}(t) = M_X(t)M_Y(t)$
- **Notes:**
  - Analogous properties hold for three or more random variables; e.g., if $X_1, \dots, X_n$ are *mutually* independent, then $E(X_1 \dots X_n) = E(X_1) \dots E(X_n)$.
  - Note that the product formula for mgf's involves the *sum* of two independent r.v.'s, not the product. The reason behind this is that the definition of the mgf of $X + Y$ is the expectation of $e^{t(X+Y)}$, which is equal to the product $e^{tX} \cdot e^{tY}$. In case of indepedence, the expectation of that product is the product of the expectations.
  - While a variance is always nonnegative, covariance and correlation can take negative values.

Practice problems (all from past actuarial exams)

1. Suppose that the cost of maintaining a car is given by a random variable, X, with mean 200 and variance 260. If a tax of 20% is introducted on all items associated with the maintenance of the car, what will the variance of the cost of maintaining a car be?

2. The profit for a new product is given by $Z = 3X - Y - 5$, where X and Y are independent random variables with $\text{Var}(X) = 1$ and $\text{Var}(Y) = 2$. What is the variance of Z?

3. An insurance policy pays a total medical benefit consisting of a part paid to the surgeon, X, and a part paid to the hospital, Y, so that the total benefit is $X + Y$. Suppose that $\text{Var}(X) = 5,000$, $\text{Var}(Y) = 10,000$, and $\text{Var}(X + Y) = 17,000$.

   If X is increased by a flat amount of 100, and Y is increased by 10%, what is the variance of the total benefit after these increases?

4. A company insures homes in three cities, J, K, L. The losses occurring in these cities are independent. The moment-generating functions for the loss distributions of the cities are

   $$M_J(t) = (1 - 2t)^{-3}, \quad M_K(t) = (1 - 2t)^{-2.5}, \quad M_L(t) = (1 - 2t)^{-4.5}$$

   Let X represent the combined losses from the three cities. Calculate $E(X^3)$.

5. Given that $E(X) = 5$, $E(X^2) = 27.4$, $E(Y) = 7$, $E(Y^2) = 51.4$ and $\text{Var}(X + Y) = 8$, find $\text{Cov}(X + Y, X + 1.2Y)$.