STAT 400 Fall 2017

Week 08 Discussion

1. Suppose that number of accidents at a construction site follows a Poisson process with the average rate of 0.80 accidents per month. Assume all months are independent of each other.

a) Find the probability that the first accident of a calendar year would occur during March.

b) Find the probability that the third accident of a calendar year would occur during April.

c) Find the probability that the third accident of a calendar year would occur during spring (March, April, or May).

"Hint": If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_{T_\alpha}(t) = P(T_\alpha \le t) = P(X_t \ge \alpha)$ and $P(T_\alpha > t) = P(X_t \le \alpha - 1)$, where X_t has a Poisson(λt) distribution.

2. As Alex is leaving for college, his parents give him a car, but warn him that they would take the car away if Alex gets 6 speeding tickets. Suppose that Alex receives speeding tickets according to Poisson process with the average rate of one ticket per six months.

a) Find the probability that it would take Alex longer than two years to get his sixth speeding ticket.

b) Find the probability that it would take Alex less than four years to get his sixth speeding ticket.

c) Find the probability that Alex would get his sixth speeding ticket during the fourth year.

d) Find the probability that Alex would get his sixth speeding ticket during the third year.

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = C(x+2y), \quad 0<x<2, \; 0<y<3, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0<x<2, \; 0<y<3 \}$.

b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

4. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = C x^2 y, \quad 0<x<4, \; 0<y<\sqrt{x}, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0<x<4, \; 0<y<\sqrt{x} \}$.

b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

e)* Are X and Y independent?
If X and Y are not independent, find $\text{Cov}(X, Y)$.

5. Let the joint probability density function for (X, Y) be

$$f(x,y) = x+y, \quad x>0, \quad y>0, \quad x+2y<2, \qquad \text{zero otherwise.}$$

a) Find the probability $P(Y>X)$.

b) Find the marginal p.d.f. of X, $f_X(x)$.

c) Find the marginal p.d.f. of Y, $f_Y(y)$.

d)* Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

6 – 9. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{12}{5} x y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

Do NOT use a computer. You may only use +, –, ×, ÷, and $\sqrt{}$ on a calculator. Show all work. Example:

$$\int_0^1 \left(\int_y^2 \frac{12}{5} x y^3 \, dx \right) dy = \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Big|_{x=y}^{x=2} dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Big|_{y=0}^{y=1} = \frac{6}{5} - \frac{1}{5} = 1. \qquad \Rightarrow \qquad f(x,y) \text{ is a valid joint p.d.f.}$$

6. a) Sketch the support of (X, Y). That is, sketch $\{ 0 < y < 1, \ y < x < 2 \}$.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

d) Are X and Y independent? Justify your answer.

7. Find the probability $P(X > 2Y)$.

a) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

b) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

c) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

d) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

e) Use one of (a) – (d) to find the desired probability.

8. Find the probability $P(X + Y < 2)$. Repeat parts (a) – (e) of problem 7.

9. Find the probability $P(XY < 1)$. Repeat parts (a) – (e) of problem 7.

10. Let the joint probability density function for (X, Y) be

$$f(x,y) = C\,x\,y, \qquad x>0, \quad y>0, \quad x^2+(y+3)^2<25,$$
zero elsewhere.

a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

b) Find $P(2X+Y>2)$.

c) Find $P(X-3Y>0)$.

11. Suppose that (X, Y) is uniformly distributed over the region defined by $x \ge 0$, $y \ge 0$, $x^2+y^2 \le 1$. That is,

$$f(x,y) = C, \qquad x \ge 0, \; y \ge 0, \; x^2+y^2 \le 1, \qquad \text{zero elsewhere.}$$

a) What is the joint probability density function of X and Y? That is, find the value of C so that $f(x,y)$ is a valid joint p.d.f.

b) Find $P(X+Y<1)$.

c) Find $P(Y>2X)$.

d)* Are X and Y independent?

12. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{C}{(2x+y)^3}, \qquad y>1, \; 0<x<y, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ y>1, \; 0<x<y \}$.

b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

e) Find $P(X+Y<2)$.

f) Find $P(X+Y>5)$.

g) Find $P(Y>3X)$.

h)* Are X and Y independent?

Answers:

1. Suppose that number of accidents at a construction site follows a Poisson process with the average rate of 0.80 accidents per month. Assume all months are independent of each other.

"Hint": If T_α has a $\text{Gamma}(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then $F_{T_\alpha}(t) = P(T_\alpha \le t) = P(X_t \ge \alpha)$ and $P(T_\alpha > t) = P(X_t \le \alpha - 1)$, where X_t has a $\text{Poisson}(\lambda t)$ distribution.

a) Find the probability that the first accident of a calendar year would occur during March.

T_1 has Exponential distribution with $\lambda = 0.80$ or $\theta = 1/0.80 = 1.25$.

$$P(2 < T_1 < 3) = \int_2^3 0.80\, e^{-0.80t}\, dt = e^{-1.60} - e^{-2.40} \approx \mathbf{0.1112}.$$

OR

$$P(2 < T_1 < 3) = P(T_1 > 2) - P(T_1 > 3) = P(X_2 = 0) - P(X_3 = 0)$$
$$= P(\text{Poisson}(1.60) = 0) - P(\text{Poisson}(2.40) = 0) = 0.202 - 0.091 = \mathbf{0.111}.$$

OR

$$P\left(\begin{array}{c}\text{no accidents during}\\ \text{the first two months}\end{array} \text{ AND } \begin{array}{c}\text{at least one accident}\\ \text{during the third month}\end{array}\right)$$
$$= P(X_2 = 0) \times P(X_1 \ge 1) = 0.202 \times (1 - 0.449) \approx \mathbf{0.111}.$$

OR

Jan	Feb	Mar	
no accident	no accident	accident(s)	
0.449 ×	0.449 ×	0.551	≈ **0.111**.

b) Find the probability that the third accident of a calendar year would occur during April.

T_3 has Gamma distribution with $\alpha = 3$ and $\lambda = 0.80$ or $\theta = {}^1\!/_{0.80} = 1.25$.

$$P(3 < T_3 < 4) = P(T_3 > 3) - P(T_3 > 4) = P(X_3 \le 2) - P(X_4 \le 2)$$
$$= P(\text{Poisson}(2.4) \le 2) - P(\text{Poisson}(3.2) \le 2) = 0.570 - 0.380 = \mathbf{0.190}.$$

OR

$$P(3 < T_3 < 4) = \int_3^4 \frac{0.80^3}{\Gamma(3)} t^{3-1} e^{-0.80t}\, dt = \int_3^4 \frac{0.80^3}{2} t^{3-1} e^{-0.80t}\, dt \approx \mathbf{0.189805}.$$

OR

$$P\left(\begin{array}{c}\text{two accidents during}\\ \text{the first three months}\end{array} \text{ AND } \begin{array}{c}\text{at least one accident}\\ \text{during April}\end{array} \right)$$
$$+ P\left(\begin{array}{c}\text{one accident during}\\ \text{the first three months}\end{array} \text{ AND } \begin{array}{c}\text{at least two accidents}\\ \text{during April}\end{array} \right)$$
$$+ P\left(\begin{array}{c}\text{no accidents during}\\ \text{the first three months}\end{array} \text{ AND } \begin{array}{c}\text{at least three accidents}\\ \text{during April}\end{array} \right) = \ldots$$

c) Find the probability that the third accident of a calendar year would occur during spring (March, April, or May).

T_3 has Gamma distribution with $\alpha = 3$ and $\lambda = 0.80$ or $\theta = {}^1/_{0.80} = 1.25$.

$$P(2 < T_3 < 5) = P(T_3 > 2) - P(T_3 > 5) = P(X_2 \le 2) - P(X_5 \le 2)$$

$$= P(\text{Poisson}(1.6) \le 2) - P(\text{Poisson}(4.0) \le 2) = 0.783 - 0.238 = \mathbf{0.545}.$$

OR

$$P(2 < T_3 < 5) = \int_2^5 \frac{0.80^3}{\Gamma(3)} t^{3-1} e^{-0.80t}\, dt = \int_2^5 \frac{0.80^3}{2} t^{3-1} e^{-0.80t}\, dt \approx \mathbf{0.545255}.$$

OR

$$P\begin{pmatrix} \text{two accidents during} & \text{AND} & \text{at least one accident} \\ \text{the first two months} & & \text{during the next three months} \end{pmatrix}$$
$$+ \; P\begin{pmatrix} \text{one accident during} & \text{AND} & \text{at least two accidents} \\ \text{the first two months} & & \text{during the next three months} \end{pmatrix}$$
$$+ \; P\begin{pmatrix} \text{no accidents during} & \text{AND} & \text{at least three accidents} \\ \text{the first two months} & & \text{during the next three months} \end{pmatrix} = \ldots$$

2. As Alex is leaving for college, his parents give him a car, but warn him that they would take the car away if Alex gets 6 speeding tickets. Suppose that Alex receives speeding tickets according to Poisson process with the average rate of one ticket per six months.

X_t = number of speeding tickets in t years. Poisson(λt)

T_k = time of the kth speeding ticket. Gamma, $\alpha = k$.

one ticket per six months $\Rightarrow$ $\lambda = 2$.

If T_α has a Gamma$(\alpha, \theta = {}^1\!/_\lambda)$ distribution, where α is an integer, then

$$P(T_\alpha \le t) = P(X_t \ge \alpha) \quad \text{and} \quad P(T_\alpha > t) = P(X_t \le \alpha - 1),$$

where X_t has a Poisson(λt) distribution.

a) Find the probability that it would take Alex longer than two years to get his sixth speeding ticket.

$$P(T_6 > 2) = P(X_2 \le 5) = P(\text{Poisson}(4) \le 5) = \mathbf{0.785}.$$

OR

$$P(T_6 > 2) = \int_2^\infty \frac{2^6}{\Gamma(6)}\, t^{6-1} e^{-2t}\, dt = \int_2^\infty \frac{2^6}{5!}\, t^5 e^{-2t}\, dt = \ldots$$

b) Find the probability that it would take Alex less than four years to get his sixth speeding ticket.

$$P(T_6 < 4) = P(X_4 \ge 6) = P(X_4 \ge 6) = 1 - P(X_4 \le 5)$$

$$= 1 - P(\text{Poisson}(8) \le 5) = 1 - 0.191 = \mathbf{0.809}.$$

OR

$$P(T_6 < 4) = \int_0^4 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_0^4 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

c) Find the probability that Alex would get his sixth speeding ticket during the fourth year.

$$P(3 < T_6 < 4) = P(T_6 > 3) - P(T_6 > 4) = P(X_3 \le 5) - P(X_4 \le 5)$$
$$= P(\text{Poisson}(6) \le 5) - P(\text{Poisson}(8) \le 5) = 0.446 - 0.191 = \mathbf{0.255}.$$

OR

$$P(3 < T_6 < 4) = \int_3^4 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_3^4 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

d) Find the probability that Alex would get his sixth speeding ticket during the third year.

$$P(2 < T_6 < 3) = P(T_6 > 2) - P(T_6 > 3) = P(X_2 \le 5) - P(X_3 \le 5)$$
$$= P(\text{Poisson}(4) \le 5) - P(\text{Poisson}(6) \le 5) = 0.785 - 0.446 = \mathbf{0.339}.$$

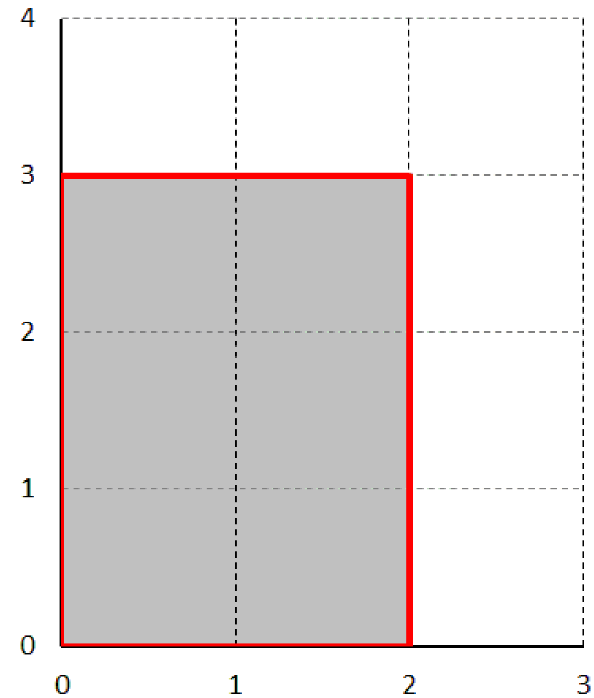
OR

$$P(2 < T_6 < 3) = \int_2^3 \frac{2^6}{\Gamma(6)} t^{6-1} e^{-2t}\, dt = \int_2^3 \frac{2^6}{5!} t^5 e^{-2t}\, dt = \ldots$$

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = C\,(x+2y), \qquad 0<x<2, \quad 0<y<3, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y).
That is, sketch
$\{\, 0<x<2, \quad 0<y<3 \,\}$.



b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^2 \left(\int_0^3 C\,(x+2y)\,dy \right) dx$$

$$= \int_0^2 \left(C\,\left(xy+y^2\right) \Big|_0^3 \right) dx$$

$$= \int_0^2 C\,(3x+9)\,dx$$

$$= C\left(\frac{3}{2}x^2+9x \right) \Big|_0^2 = 24\,C. \qquad \Rightarrow \qquad C = \mathbf{\frac{1}{24}}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^3 \frac{1}{24}(x+2y)\,dy = \frac{1}{24}\left(xy+y^2\right)\Big|_0^3 = \frac{1}{24}(3x+9) = \frac{x+3}{8}, \qquad 0<x<2.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

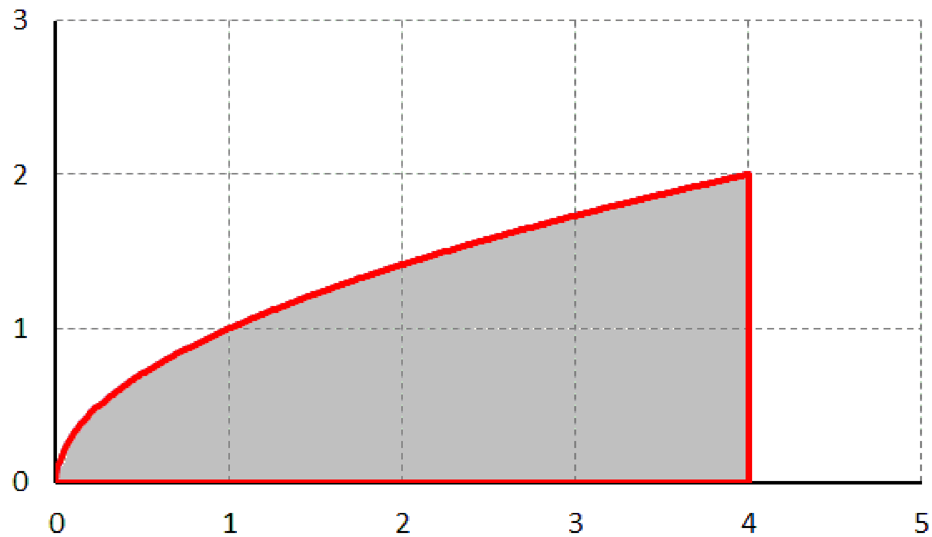
$$f_Y(y) = \int_0^2 \frac{1}{24}(x+2y)\,dx = \frac{1}{24}\left(\frac{x^2}{2}+2xy \right)\Big|_0^2$$

$$= \frac{1}{24}(2+4y) = \frac{1+2y}{12}, \qquad 0<y<3.$$

4. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = C\,x^2 y, \qquad 0 < x < 4, \quad 0 < y < \sqrt{x}, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{\, 0 < x < 4,\ 0 < y < \sqrt{x} \,\}$.



b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

$$1 = \int_0^4 \left(\int_0^{\sqrt{x}} C\,x^2\,y\,dy \right) dx = \int_0^4 \frac{C}{2}\,x^2\,y^2 \Big|_0^{\sqrt{x}} dx = \int_0^4 \frac{C}{2}\,x^3\,dx = \frac{C}{8}\,x^4 \Big|_0^4 = 32\,C.$$

$$\Rightarrow \qquad C = \mathbf{\frac{1}{32}}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^{\sqrt{x}} \frac{1}{32}\,x^2\,y\,dy = \frac{1}{64}\,x^2\,y^2 \Big|_0^{\sqrt{x}} = \frac{1}{64}\,x^3, \qquad 0 < x < 4.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_{y^2}^{4} \frac{1}{32} x^2 y \, dx = \frac{y}{96} \cdot x^3 \Big|_{y^2}^{4} = \frac{y}{96} \cdot \left(64 - y^6\right) = \frac{2}{3} y - \frac{1}{96} y^7, \qquad 0 < y < 2.$$

e) Are X and Y independent?
If X and Y are not independent, find $\mathrm{Cov}(X, Y)$.

$f(x,y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \int_0^4 x \cdot \frac{1}{64} x^3 \, dx = \int_0^4 \frac{1}{64} x^4 \, dx = \frac{1}{320} x^5 \Big|_0^4 = \frac{16}{5} = 3.2.$$

$$E(Y) = \int_0^2 y \cdot \left(\frac{2}{3} y - \frac{1}{96} y^7 \right) dy = \int_0^2 \left(\frac{2}{3} y^2 - \frac{1}{96} y^8 \right) dy = \left(\frac{2}{9} y^3 - \frac{1}{864} y^9 \right) \Big|_0^2$$

$$= \frac{16}{9} - \frac{16}{27} = \frac{32}{27} \approx 1.1852.$$

$$E(XY) = \int_0^4 \left(\int_0^{\sqrt{x}} xy \cdot \frac{1}{32} x^2 y \, dy \right) dx = \int_0^4 \frac{1}{96} x^3 y^3 \Big|_0^{\sqrt{x}} dx = \int_0^4 \frac{1}{96} x^{9/2} \, dx$$

$$= \frac{1}{528} x^{11/2} \Big|_0^4 = \frac{128}{33} \approx 3.8788.$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{128}{33} - \frac{16}{5} \cdot \frac{32}{27} = \mathbf{\frac{128}{1485}} \approx 0.0862.$$

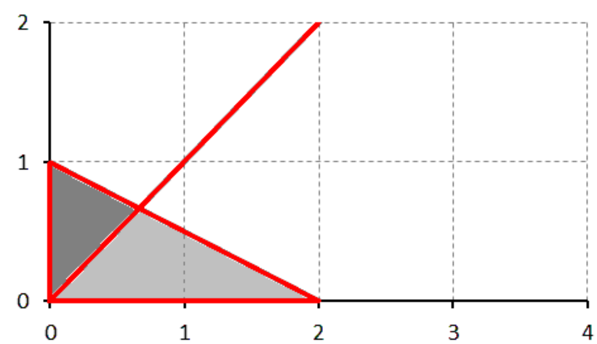
5. Let the joint probability density function for (X, Y) be

$$f(x,y) = x + y,$$

$$x > 0, \quad y > 0, \quad x + 2y < 2,$$

zero otherwise.

a) Find the probability $P(Y > X)$.



intersection point:

$y = x$ and $x + 2y = 2$

$x = \frac{2}{3}$ and $y = \frac{2}{3}$

$$P(Y > X) = \int_0^{2/3} \left(\int_x^{1-(x/2)} (x+y)\,dy \right) dx = \int_0^{2/3} \left(\frac{1}{2} + \frac{1}{2}x - \frac{15}{8}x^2 \right) dx = \mathbf{\frac{7}{27}}.$$

OR

$$P(Y > X) = 1 - \int_0^{2/3} \left(\int_y^{2-2y} (x+y)\,dx \right) dy = 1 - \int_0^{2/3} \left(2 - 2y - \frac{3}{2}y^2 \right) dx = \mathbf{\frac{7}{27}}.$$

b) Find the marginal p.d.f. of X, $f_X(x)$.

$$f_X(x) = \int_0^{1-(x/2)} (x+y)\,dy = \frac{1}{2} + \frac{1}{2}x - \frac{3}{8}x^2, \qquad 0 < x < 2.$$

c) Find the marginal p.d.f. of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^{2-2y} (x+y)\,dx = 2-2y, \qquad 0 < y < 1.$$

d)* Are X and Y independent? If not, find $\mathrm{Cov}(X, Y)$.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f_{X,Y}(x, y) \neq f_X(x) \times f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^2 x \cdot \left(\frac{1}{2} + \frac{1}{2}x - \frac{3}{8}x^2\right) dx = \int_0^2 \left(\frac{1}{2}x + \frac{1}{2}x^2 - \frac{3}{8}x^3\right) dx$$

$$= \left(\frac{1}{4}x^2 + \frac{1}{6}x^3 - \frac{3}{32}x^4\right)\Bigg|_0^2 = 1 + \frac{4}{3} - \frac{3}{2} = \frac{5}{6}.$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot (2-2y)\,dy = \int_0^1 \left(2y - 2y^2\right) dy$$

$$= \left(y^2 - \frac{2}{3}y^3\right)\Bigg|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$E(XY) = \int_0^1 \left(\int_0^{2-2y} xy \cdot (x+y)\,dx\right) dy = \int_0^1 \left(\frac{y}{3}(2-2y)^3 + \frac{y^2}{2}(2-2y)^2\right) dx$$

$$= \int_0^1 \left(\frac{8}{3}y - 6y^2 + 4y^3 - \frac{2}{3}y^4\right) dy = \frac{4}{3} - 2 + 1 - \frac{2}{15} = \frac{1}{5}.$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \times E(Y) = \frac{1}{5} - \frac{5}{6} \times \frac{1}{3} = -\mathbf{\frac{7}{90}} \approx -0.077778.$$

6 – 9. Let the joint probability density function for (X, Y) be

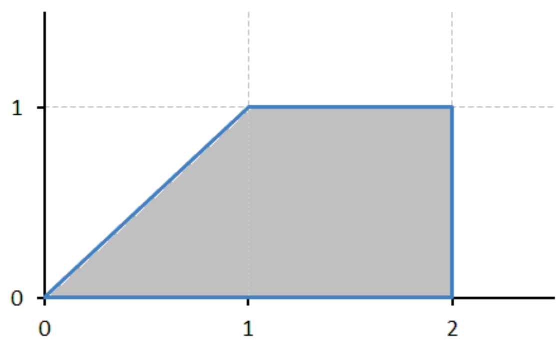
$$f(x,y) = \frac{12}{5} x y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

Do NOT use a computer. You may only use +, –, ×, ÷, and $\sqrt{\ }$ on a calculator. Show all work. Example:

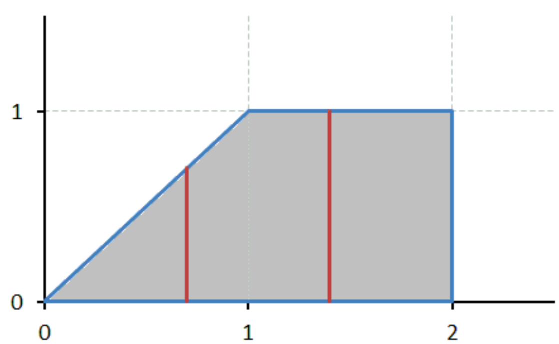
$$\int_0^1 \left(\int_y^2 \frac{12}{5} x y^3 \, dx \right) dy = \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Bigg|_{x=y}^{x=2} dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = \frac{6}{5} - \frac{1}{5} = 1. \qquad \Rightarrow \qquad f(x,y) \text{ is a valid joint p.d.f.}$$

6. a) Sketch the support of (X, Y). That is, sketch $\{ 0 < y < 1, \; y < x < 2 \}$.



b) Find the marginal probability density function of X, $f_X(x)$.

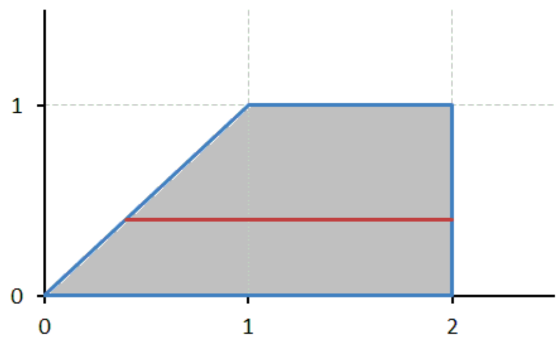


For $0 < x < 1$, $\quad f_X(x) = \int_0^x \frac{12}{5} x y^3 \, dy = \left(\frac{3}{5} x y^4 \right) \Big|_{y=0}^{y=x} = \frac{3}{5} x^5.$

For $1 < x < 2$, $\quad f_X(x) = \int_0^1 \frac{12}{5} x y^3 \, dy = \left(\frac{3}{5} x y^4 \right) \Big|_{y=0}^{y=1} = \frac{3}{5} x.$

Check: $$\int_0^1 \frac{3}{5}x^5\,dx + \int_1^2 \frac{3}{5}x\,dx = \left(\frac{1}{10}x^6\right)\Bigg|_{x=0}^{x=1} + \left(\frac{3}{10}x^2\right)\Bigg|_{x=1}^{x=2} = \frac{1}{10} + \frac{9}{10} = 1.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.



For $0 < y < 1$, $$f_Y(y) = \int_y^2 \frac{12}{5}x\,y^3\,dx = \left(\frac{6}{5}x^2\,y^3\right)\Bigg|_{x=y}^{x=2} = \frac{24}{5}y^3 - \frac{6}{5}y^5.$$

Check: $$\int_0^1 \left(\frac{24}{5}y^3 - \frac{6}{5}y^5\right)dy = \left(\frac{6}{5}y^4 - \frac{1}{5}y^6\right)\Bigg|_{y=0}^{y=1} = \frac{6}{5} - \frac{1}{5} = 1.$$

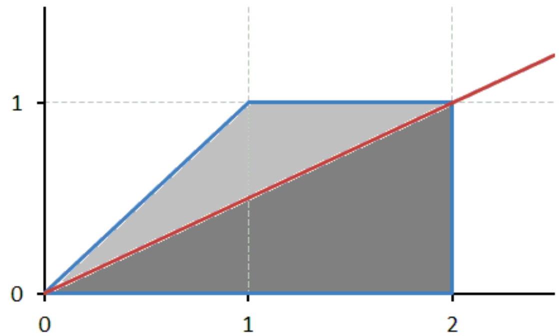
d) Are X and Y independent? Justify your answer.

The support of (X, Y) is NOT a rectangle. X and Y are **NOT independent**.

OR

Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$, X and Y are **NOT independent**.

7. Find the probability $P(X > 2Y)$.



a) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^2 \left(\int_0^{x/2} \frac{12}{5} x y^3 \, dy \right) dx$$

b) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1 \left(\int_{2y}^{2} \frac{12}{5} x y^3 \, dx \right) dy$$

c) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^1 \left(\int_{x/2}^{x} \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_{x/2}^{1} \frac{12}{5} x y^3 \, dy \right) dx$$

d) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1 \left(\int_{y}^{2y} \frac{12}{5} x y^3 \, dx \right) dy$$

e) Use one of (a) – (d) to find the desired probability.

(a) $$\int_0^2 \left(\int_0^{x/2} \frac{12}{5} x y^3 \, dy \right) dx = \int_0^2 \frac{3}{80} x^5 \, dx = \left(\frac{1}{160} x^6 \right) \Bigg|_{x=0}^{x=2} = \frac{\mathbf{2}}{\mathbf{5}} = \mathbf{0.40}.$$

(b) $$\int_0^1 \left(\int_{2y}^{2} \frac{12}{5} x y^3 \, dx \right) dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{24}{5} y^5 \right) dy = \left(\frac{6}{5} y^4 - \frac{4}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = \frac{\mathbf{2}}{\mathbf{5}}.$$

(c) $$1 - \int_0^1 \left(\int_{x/2}^{x} \frac{12}{5} x y^3 \, dy \right) dx - \int_1^2 \left(\int_{x/2}^{1} \frac{12}{5} x y^3 \, dy \right) dx$$

$$= 1 - \int_0^1 \left(\frac{3}{5} x^5 - \frac{3}{80} x^5 \right) dx - \int_1^2 \left(\frac{3}{5} x - \frac{3}{80} x^5 \right) dx$$

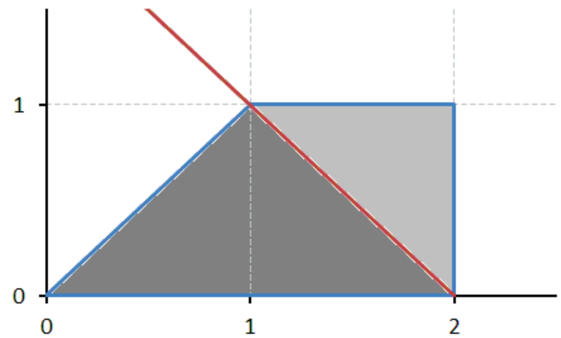
$$= 1 - \left(\frac{1}{10} x^6 - \frac{1}{160} x^6 \right) \Bigg|_{x=0}^{x=1} - \left(\frac{3}{10} x^2 - \frac{1}{160} x^6 \right) \Bigg|_{x=1}^{x=2}$$

$$= 1 - \left(\frac{1}{10} - \frac{1}{160} \right) - \left(\frac{6}{5} - \frac{2}{5} \right) + \left(\frac{3}{10} - \frac{1}{160} \right) = \frac{\mathbf{2}}{\mathbf{5}}.$$

(d) $$1 - \int_0^1 \left(\int_{y}^{2y} \frac{12}{5} x y^3 \, dx \right) dy = 1 - \int_0^1 \left(\frac{24}{5} y^5 - \frac{6}{5} y^5 \right) dy = 1 - \int_0^1 \frac{18}{5} y^5 \, dy$$

$$= 1 - \left(\frac{3}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = 1 - \frac{3}{5} = \frac{\mathbf{2}}{\mathbf{5}}.$$

8. Find the probability $P(X+Y<2)$.



a) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^1 \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_0^{2-x} \frac{12}{5} x y^3 \, dy \right) dx$$

b) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1 \left(\int_y^{2-y} \frac{12}{5} x y^3 \, dx \right) dy$$

c) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_1^2 \left(\int_{2-x}^{1} \frac{12}{5} x y^3 \, dy \right) dx$$

d) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^1 \left(\int_{2-y}^{2} \frac{12}{5} x y^3 \, dx \right) dy$$

e) Use one of (a) – (d) to find the desired probability.

(b) $$\int_0^1\left(\int_y^{2-y}\frac{12}{5}xy^3\,dx\right)dy = \int_0^1\left(\frac{6}{5}x^2y^3\right)\Bigg|_{x=y}^{x=2-y}dy = \int_0^1\left(\frac{24}{5}y^3-\frac{24}{5}y^4\right)dy$$

$$= \left(\frac{6}{5}y^4-\frac{24}{25}y^5\right)\Bigg|_{y=0}^{y=1} = \mathbf{\frac{6}{25}} = \mathbf{0.24}.$$

(d) $$1-\int_0^1\left(\int_{2-y}^{2}\frac{12}{5}xy^3\,dx\right)dy = 1-\int_0^1\left(\frac{6}{5}x^2y^3\right)\Bigg|_{x=2-y}^{x=2}dy$$

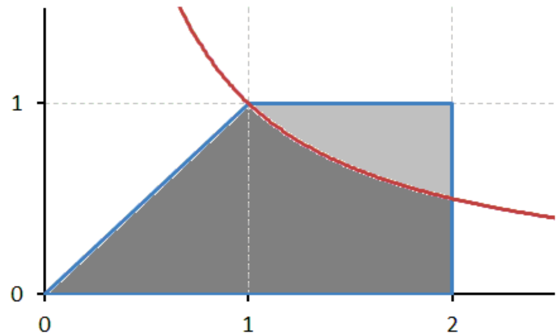
$$= 1-\int_0^1\left(\frac{24}{5}y^4-\frac{6}{5}y^5\right)dy = 1-\left(\frac{24}{25}y^5-\frac{1}{5}y^6\right)\Bigg|_{y=0}^{y=1} = \mathbf{\frac{6}{25}}.$$

(a) $$\int_0^1\left(\int_0^x\frac{12}{5}xy^3\,dy\right)dx + \int_1^2\left(\int_0^{2-x}\frac{12}{5}xy^3\,dy\right)dx$$

$$= \int_0^1\frac{3}{5}x^5\,dx + \int_1^2\frac{3}{5}x(2-x)^4\,dx = \frac{1}{10} + \int_1^2\frac{3}{5}x(2-x)^4\,dx = \ldots$$

(c) $$1-\int_1^2\left(\int_{2-x}^{1}\frac{12}{5}xy^3\,dy\right)dx = 1-\int_1^2\left(\frac{3}{5}x-\frac{3}{5}x(2-x)^4\right)dx = \ldots$$

9. Find the probability $P(XY < 1)$.



a) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_0^1 \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_0^{1/x} \frac{12}{5} x y^3 \, dy \right) dx$$

b) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_0^{1/2} \left(\int_y^2 \frac{12}{5} x y^3 \, dx \right) dy + \int_{1/2}^1 \left(\int_y^{1/y} \frac{12}{5} x y^3 \, dx \right) dy$$

c) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

$$\int_1^2 \left(\int_{1/x}^1 \frac{12}{5} x y^3 \, dy \right) dx$$

d) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

$$\int_{1/2}^1 \left(\int_{1/y}^2 \frac{12}{5} x y^3 \, dx \right) dy$$

e) Use one of (a) – (d) to find the desired probability.

(a) $$\int_0^1 \left(\int_0^x \frac{12}{5} x y^3 \, dy \right) dx + \int_1^2 \left(\int_0^{1/x} \frac{12}{5} x y^3 \, dy \right) dx = \int_0^1 \frac{3}{5} x^5 \, dx + \int_1^2 \frac{3}{5} \frac{1}{x^3} \, dx$$

$$= \left(\frac{1}{10} x^6 \right) \Big|_{x=0}^{x=1} + \left(-\frac{3}{10} \frac{1}{x^2} \right) \Big|_{x=1}^{x=2} = \frac{1}{10} - \frac{3}{40} + \frac{3}{10} = \mathbf{\frac{13}{40}} = \mathbf{0.325}.$$

(b) $$\int_0^{1/2} \left(\int_y^2 \frac{12}{5} x y^3 \, dx \right) dy + \int_{1/2}^1 \left(\int_y^{1/y} \frac{12}{5} x y^3 \, dx \right) dy$$

$$= \int_0^{1/2} \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy + \int_{1/2}^1 \left(\frac{6}{5} y - \frac{6}{5} y^5 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Big|_0^{1/2} + \left(\frac{3}{5} y^2 - \frac{1}{5} y^6 \right) \Big|_{1/2}^1$$

$$= \frac{3}{40} - \frac{1}{320} + \frac{3}{5} - \frac{1}{5} - \frac{3}{20} + \frac{1}{320} = \frac{2}{5} - \frac{3}{40} = \mathbf{\frac{13}{40}}.$$

(c) $$1 - \int_1^2 \left(\int_{1/x}^1 \frac{12}{5} x y^3 \, dy \right) dx = 1 - \int_1^2 \left(\frac{3}{5} x - \frac{3}{5} \frac{1}{x^3} \right) dx = 1 - \left(\frac{3}{10} x^2 + \frac{3}{10} \frac{1}{x^2} \right) \Big|_{x=1}^{x=2}$$

$$= 1 - \left(\frac{6}{5} + \frac{3}{40} - \frac{3}{10} - \frac{3}{10} \right) = 1 - \left(\frac{3}{5} + \frac{3}{40} \right) = 1 - \frac{27}{40} = \mathbf{\frac{13}{40}}.$$

(d) $$1 - \int_{1/2}^1 \left(\int_{1/y}^2 \frac{12}{5} x y^3 \, dx \right) dy = 1 - \int_{1/2}^1 \left(\frac{24}{5} y^3 - \frac{6}{5} y \right) dy = 1 - \left(\frac{6}{5} y^4 - \frac{3}{5} y^2 \right) \Big|_{1/2}^1$$

$$= 1 - \left(\frac{6}{5} - \frac{3}{5} - \frac{3}{40} + \frac{3}{20} \right) = 1 - \left(\frac{3}{5} + \frac{3}{40} \right) = 1 - \frac{27}{40} = \mathbf{\frac{13}{40}}.$$

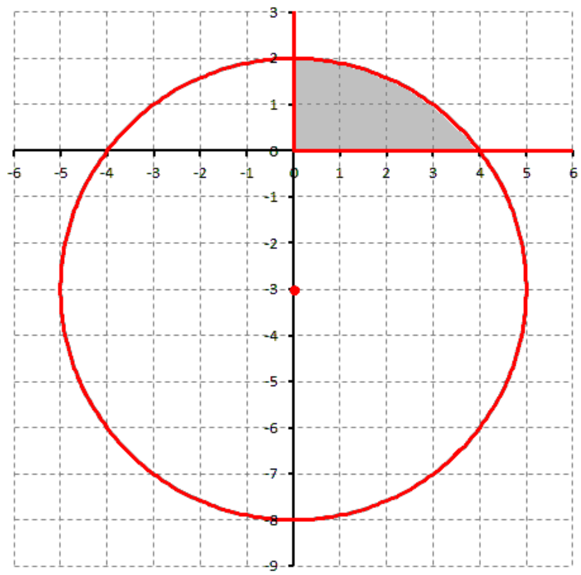
10. Let the joint probability density function for (X, Y) be

$$f(x,y) = C\,x\,y,$$

$$x > 0, \qquad y > 0,$$

$$x^2 + (y+3)^2 < 25,$$

zero elsewhere.



a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

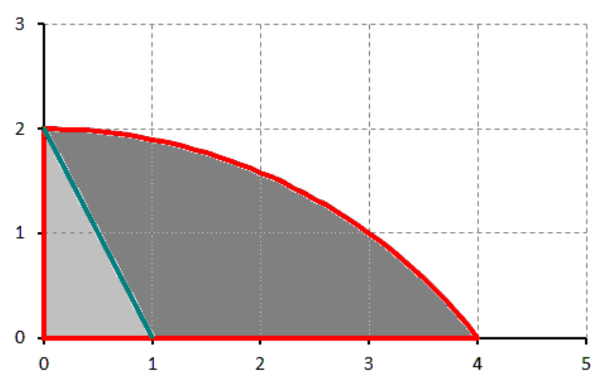
Must have
$$1 = \int_0^2 \left[\int_0^{\sqrt{25-(y+3)^2}} C\,x\,y\,dx \right] dy = \int_0^2 \frac{C}{2}\,y\left[25-(y+3)^2\right]dy$$

$$= \frac{C}{2}\int_0^2 \left[16\,y - 6\,y^2 - y^3\right]dy$$

$$= \frac{C}{2}\left[8\,y^2 - 2\,y^3 - \frac{1}{4}y^4\right]\Bigg|_0^2 = 6\,C.$$

$\Rightarrow \qquad C = \mathbf{\frac{1}{6}}.$

b) Find $P(2X + Y > 2)$.



$$1 - \int_0^1 \left(\int_0^{2-2x} \frac{1}{6}\,x\,y\,dy \right) dx$$

$$= 1 - \int_0^1 \frac{1}{12}(2-2x)^2\,x\,dx$$

$$= 1 - \int_0^1 \frac{1}{3}\left(x - 2x^2 + x^3\right)dx = 1 - \frac{1}{3}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 1 - \frac{1}{36} = \mathbf{\frac{35}{36}}.$$

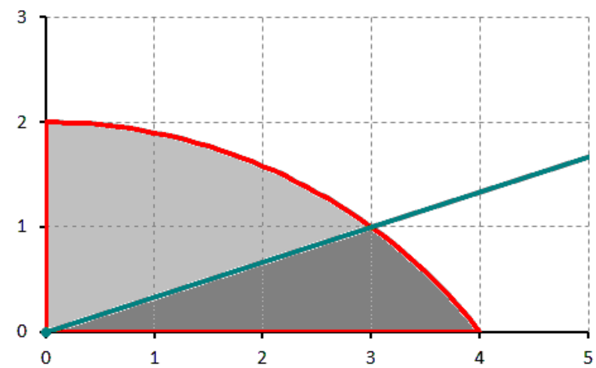
OR

$$1-\int_0^2\left(\int_0^{\frac{2-y}{2}}\frac{1}{6}\,x\,y\,dx\right)dy \;=\; \ldots \qquad\qquad \int_0^2\left(\int_{\frac{2-y}{2}}^{\sqrt{25-(y+3)^2}}\frac{1}{6}\,x\,y\,dx\right)dy \;=\; \ldots$$

OR

$$\int_0^1\left(\int_{2-2x}^{-3+\sqrt{25-x^2}}\frac{1}{6}\,x\,y\,dy\right)dx+\int_1^4\left(\int_0^{-3+\sqrt{25-x^2}}\frac{1}{6}\,x\,y\,dy\right)dx \;=\; \ldots$$

c) Find $P(X-3Y>0)$.



$$P(X-3Y>0) \;=\; P(X>3Y)$$

$$= \int_0^1\left[\int_{3y}^{\sqrt{25-(y+3)^2}}\frac{1}{6}\,x\,y\,dx\right]dy$$

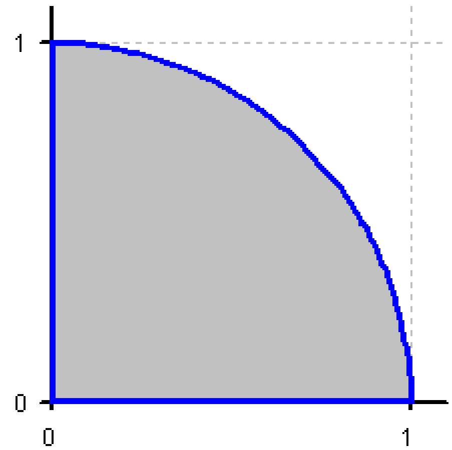
$$= \int_0^1 \frac{1}{12}\,y\left[25-(y+3)^2-9y^2\right]dy \;=\; \int_0^1 \frac{1}{12}\,y\left[16-6y-10y^2\right]dy$$

$$= \int_0^1\left[\frac{4}{3}y-\frac{1}{2}y^2-\frac{5}{6}y^3\right]dy \;=\; \frac{2}{3}-\frac{1}{6}-\frac{5}{24} \;=\; \mathbf{\frac{7}{24}} \;\approx\; 0.2916667.$$

11. Suppose that (X, Y) is uniformly distributed over the region defined by $x \ge 0,\ \ y \ge 0,\ \ x^2 + y^2 \le 1$. That is,

$$f(x, y) = C, \qquad x \ge 0,\ \ y \ge 0,\ \ x^2 + y^2 \le 1, \qquad\qquad \text{zero elsewhere.}$$

a) What is the joint probability density function of X and Y? That is, find the value of C so that $f(x, y)$ is a valid joint p.d.f.

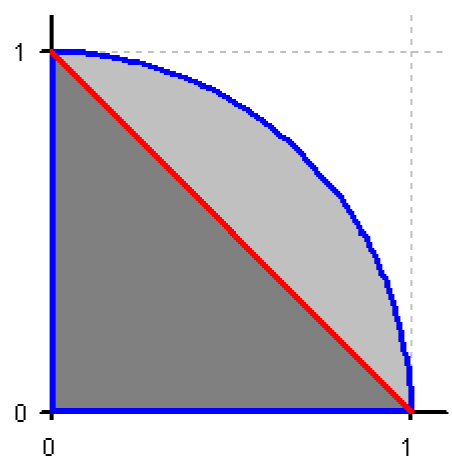


The area of a circle is πr^2.

$\Rightarrow$ The area of the support of (X, Y) is $\dfrac{\pi}{4}$.

$$\Rightarrow \qquad C = \frac{\mathbf{4}}{\boldsymbol{\pi}} \approx 1.27324.$$

b) Find $P(X + Y < 1)$.

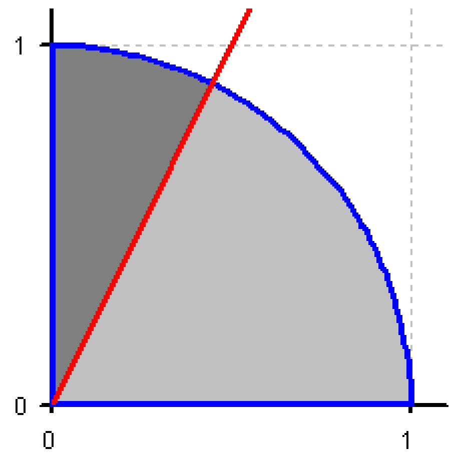


Since uniform,

$$\frac{want\ \ area}{total\ \ area} = \frac{\frac{1}{2}}{\frac{\pi}{4}} = \frac{2}{\pi}$$

$$\approx 0.63662.$$

c) Find $P(Y > 2X)$.



Since uniform,

$$1 - \frac{\arctan(2)}{\frac{\pi}{2}} = 1 - \frac{2 \times \arctan(2)}{\pi}$$

$$\approx 0.295167.$$

OR

$$\frac{\arctan\left(\frac{1}{2}\right)}{\frac{\pi}{2}} = \frac{2 \times \arctan\left(\frac{1}{2}\right)}{\pi}$$

$$\approx 0.295167.$$

d)* Are X and Y independent?

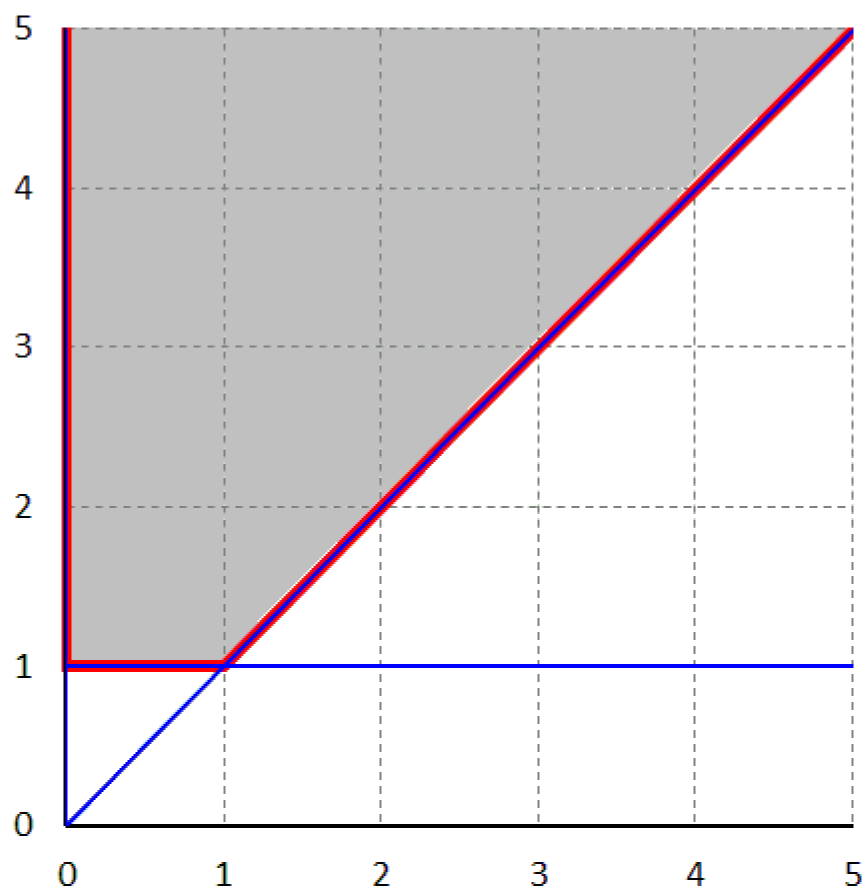
The support of (X, Y) is not a rectangle.

$\Rightarrow$ X and Y are **NOT independent**.

12. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \frac{C}{(2x+y)^3}, \qquad y > 1, \quad 0 < x < y, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ y > 1, \ 0 < x < y \}$.



b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

$$1 = \int_1^\infty \left(\int_0^y \frac{C}{(2x+y)^3}\,dx \right) dy = \int_1^\infty \left(-\frac{C}{4(2x+y)^2} \right) \Bigg|_0^y dy$$

$$= \int_1^\infty \left(-\frac{C}{36y^2} + \frac{C}{4y^2} \right) dy = \frac{2C}{9} \int_1^\infty \frac{1}{y^2}\,dy = \frac{2C}{9}\left(-\frac{1}{y} \right) \Bigg|_1^\infty = \frac{2C}{9}.$$

$$\Rightarrow \quad C = \mathbf{\frac{9}{2}} = \mathbf{4.5}.$$

c) Find the marginal probability density function of X, $f_X(x)$.

For $0 < x < 1$,

$$f_X(x) = \int_1^\infty \frac{9}{2(2x+y)^3}\,dy = -\frac{9}{4(2x+y)^2} \Bigg|_1^\infty = \frac{9}{4(2x+1)^2}, \qquad 0 < x < 1.$$

For $1 < x < \infty$,

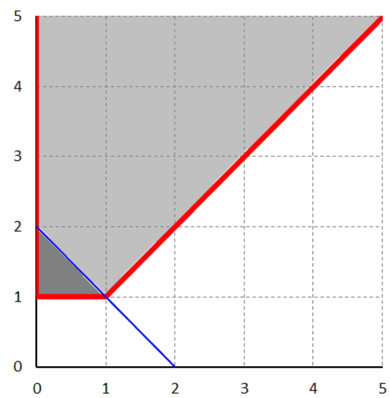
$$f_X(x) = \int_x^\infty \frac{9}{2(2x+y)^3}\,dy = -\frac{9}{4(2x+y)^2} \Bigg|_x^\infty = \frac{1}{4x^2}, \qquad 1 < x < \infty.$$

d) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^y \frac{9}{2(2x+y)^3}\,dx = -\frac{9}{8(2x+y)^2} \Bigg|_0^y$$

$$= -\frac{1}{8y^2} + \frac{9}{8y^2} = \frac{1}{y^2}, \qquad 1 < y < \infty.$$

e) Find $P(X+Y<2)$.



$$P(X+Y<2)$$

$$= \int_0^1 \left(\int_1^{2-x} \frac{9}{2(2x+y)^3}\,dy \right) dx$$

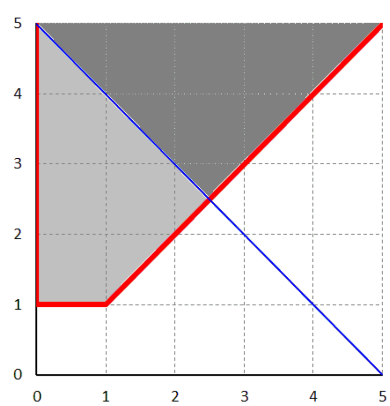
$$= \int_0^1 \left(-\frac{9}{4(2x+y)^2} \right) \Bigg|_1^{2-x} dx$$

$$= \int_0^1 \left(\frac{9}{4(2x+1)^2} - \frac{9}{4(x+2)^2} \right) dx$$

$$= \left(-\frac{9}{8(2x+1)} + \frac{9}{4(x+2)} \right) \Bigg|_0^1$$

$$= -\frac{9}{24} + \frac{9}{12} + \frac{9}{8} - \frac{9}{8} = \mathbf{\frac{3}{8}} = \mathbf{0.375}.$$

f) Find $P(X+Y>5)$.



$$P(X+Y>5)$$

$$= \int_0^{2.5} \left(\int_{5-x}^{\infty} \frac{9}{2(2x+y)^3}\,dy \right) dx$$

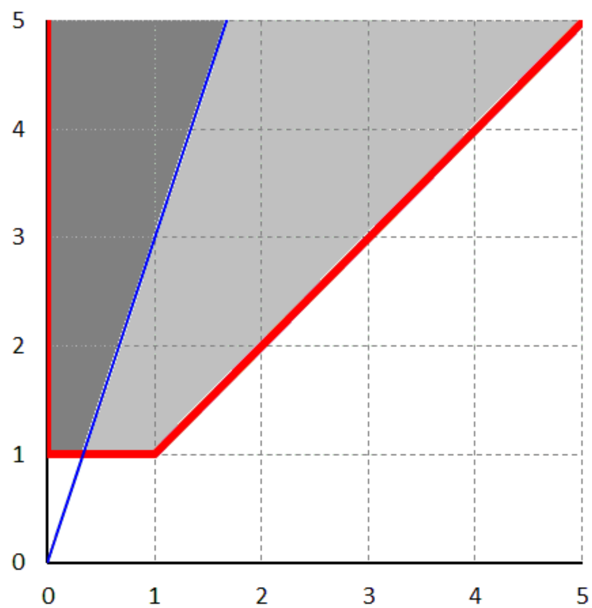
$$+ \int_{2.5}^{\infty} \left(\int_x^{\infty} \frac{9}{2(2x+y)^3}\,dy \right) dx$$

$$= \int_0^{2.5} \frac{9}{4(x+5)^2}\,dx + \int_{2.5}^{\infty} \frac{1}{4x^2}\,dx$$

$$= -\frac{9}{4(x+5)} \Bigg|_0^{2.5} - \frac{1}{4x} \Bigg|_{2.5}^{\infty}$$

$$= -\frac{9}{30} + \frac{9}{20} - 0 + \frac{1}{10} = \mathbf{0.25}.$$

g) Find $P(Y > 3X)$.



$P(Y > 3X)$

$$= \int_1^{\infty} \left(\int_0^{y/3} \frac{9}{2(2x+y)^3}\, dx \right) dy$$

$$= \int_1^{\infty} \left(-\frac{9}{8(2x+y)^2} \right) \Bigg|_0^{y/3} dy$$

$$= \int_1^{\infty} \left(\frac{9}{8y^2} - \frac{81}{200y^2} \right) dy$$

$$= \frac{9}{8} - \frac{81}{200} = \mathbf{0.72}.$$

h)* Are X and Y independent?

$f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

OR

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.