STAT 400 | Fall 2017

Week 11 Discussion

1. Let $\tau > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\tau) = \frac{\tau^5}{8} x^{14} e^{-\tau x^3}, \qquad x > 0.$$

Obtain the maximum likelihood estimator of τ, $\hat{\tau}$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = \frac{2(\theta - x)}{\theta^2} \qquad 0 < x < \theta \qquad \theta > 0.$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

b) Is $\tilde{\theta}$ an unbiased estimator for θ?

c) Find $\operatorname{Var}(\tilde{\theta})$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\lambda) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x^2}, \quad x > 0, \qquad \lambda > 0.$$

a) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

b) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find the maximum likelihood estimate of λ.

c) Obtain the method of moments estimator of λ, $\tilde{\lambda}$.

d) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find a method of moments estimate of λ.

e) Find a closed-form expression for $E(X^k)$, $k > -1$.

4. Let $\lambda > 0$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\lambda) = \frac{\lambda^2}{2}\, e^{-\lambda\sqrt{x}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Find the maximum likelihood estimator of λ, $\hat{\lambda}$.

b) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$.
Find the maximum likelihood estimate of λ, $\hat{\lambda}$.

c) Find a closed-form expression for $E(X^k)$ for $k > -1$.

"Hint" 1: $u = \sqrt{x}$.

"Hint" 2: $\dfrac{\lambda^{\alpha}}{\Gamma(\alpha)}\, u^{\alpha-1} e^{-\lambda u}$ is the p.d.f. of Gamma$(\alpha, \theta = \frac{1}{\lambda})$ distribution

d) Find $E(X)$ and $Var(X)$.

e) Find a method of moments estimator of λ, $\tilde{\lambda}$.

f) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$.
Find a method of moments estimate of λ, $\tilde{\lambda}$.

5. Let $\lambda > 0$ and let $X_1, X_2, \dots, X_n$ be independent random variables, each with the probability density function

$$f(x;\lambda) = \frac{\lambda}{x^{\lambda+1}}, \qquad x > 1.$$

a) (i) Find the maximum likelihood estimator of λ, $\hat{\lambda}$.

(ii) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$.
Find the maximum likelihood estimate of λ.

b) (i) Find a method of moments estimator of λ, $\tilde{\lambda}$. (Assume $\lambda > 1$.)

(ii) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$.

Find a method of moments estimate of λ.

6. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability mass function

$$P(X_i = 1) = \frac{\theta}{3+\theta}, \quad P(X_i = 2) = \frac{2}{3+\theta}, \quad P(X_i = 3) = \frac{1}{3+\theta}, \qquad \theta > 0.$$

a) Obtain the method of moments estimator $\tilde{\theta}$ of θ.

b) Obtain the maximum likelihood estimator $\hat{\theta}$ of θ.

7. Bert and Ernie find a coin on the sidewalk on Sesame Street. They wish to estimate p, the probability of Heads. Bert got X Heads in N coin tosses (N is fixed, X is random). Ernie got Heads for the first time on the Y^{th} coin toss (Y is random). They decide to combine their information in hope of a better estimate. (Assume independence.)

a) What is the likelihood function $L(p) = L(p; X, N, Y)$?

b) Obtain the maximum likelihood estimator for p.

c) Explain intuitively why your estimator makes good sense.

8. Let $\theta \in \mathbb{R}$ and let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x; \theta) = \frac{1}{2} e^{-|x-\theta|}, \qquad x \in \mathbb{R}.$$

a) Find a method of moments estimator $\tilde{\theta}$ of θ.

b) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

9. A random sample of size $n = 16$ from $N(\mu, \sigma^2 = 64)$ yielded $\overline{x} = 85$.
Construct the following confidence intervals for μ:

a) 95%. b) 90%. c) 80%.

10. What is the minimum sample size required for estimating μ for $N(\mu, \sigma^2 = 64)$ to within ± 3 with confidence level

a) 95%. b) 90%. c) 80%.

11. Suppose the overall (population) standard deviation of the bill amounts at a supermarket is $\sigma = \$13.75$.

a) Find the probability that the sample mean bill amount will be within \$2.00 of the overall mean bill amount for a random sample of 121 customers.

b) What is the minimum sample size required for estimating the overall mean bill amount to within \$2.00 with 95% confidence?

12. 11. (continued)
The supermarket selected a random sample of 121 customers, which showed the sample mean bill amount of \$78.80.

c) Construct a 95% confidence interval for the overall mean bill amount at this supermarket.

d) Suppose the supermarket puts Alex in charge of computing the confidence interval, and he gets the answer $(76.15, 81.45)$. Alex says that he used a different confidence level, but other than that did everything correctly. Find the confidence level used by Alex.

Answers:

1. Let $\tau > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\tau) = \frac{\tau^5}{8} x^{14} e^{-\tau x^3}, \qquad x > 0.$$

Obtain the maximum likelihood estimator of τ, $\hat{\tau}$.

$$L(\tau) = \prod_{i=1}^{n} \left(\frac{\tau^5}{8} x_i^{14} e^{-\tau x_i^3} \right) = \frac{\tau^{5n}}{8^n} \left(\prod_{i=1}^{n} x_i^{14} \right) e^{-\tau \sum_{i=1}^{n} x_i^3}$$

$$\ln L(\tau) = 5n \cdot \ln \tau - n \cdot \ln 8 + 14 \sum_{i=1}^{n} \ln x_i - \tau \cdot \sum_{i=1}^{n} x_i^3$$

$$(\ln L(\tau))' = \frac{5n}{\tau} - \sum_{i=1}^{n} x_i^3 = 0$$

$$\Rightarrow \qquad \hat{\tau} = \frac{5n}{\sum_{i=1}^{n} X_i^3}.$$

2. Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = \frac{2(\theta - x)}{\theta^2} \qquad 0 < x < \theta \qquad \theta > 0.$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\theta} x \cdot \left(\frac{2}{\theta} - \frac{2}{\theta^2} x \right) dx = \left(\frac{x^2}{\theta} - \frac{2}{3} \cdot \frac{x^3}{\theta^2} \right) \Bigg|_0^{\theta} = \frac{\theta}{3}.$$

$$\overline{X} = \frac{\tilde{\theta}}{3}. \qquad \tilde{\theta} = 3 \cdot \overline{X} = 3 \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} X_i$$

b)* Is $\tilde{\theta}$ an unbiased estimator for θ?

$$E(\tilde{\theta}) = E(3\,\overline{X}) = 3\,E(\overline{X}) = 3\,\mu = 3\,\frac{\theta}{3} = \theta.$$

$\Rightarrow$ $\tilde{\theta}$ an unbiased estimator for θ.

c)* Find $\operatorname{Var}(\tilde{\theta})$.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^{\theta} x^2 \cdot \left(\frac{2}{\theta} - \frac{2}{\theta^2} x \right) dx = \frac{\theta^2}{6}.$$

$$\sigma^2 = \operatorname{Var}(X) = \frac{\theta^2}{6} - \frac{\theta^2}{9} = \frac{\theta^2}{18}.$$

$$\operatorname{Var}(\tilde{\theta}) = \operatorname{Var}(3\,\overline{X}) = 9\,\operatorname{Var}(\overline{X}) = 9 \cdot \frac{\sigma^2}{n} = 9 \cdot \frac{\theta^2}{18n} = \frac{\theta^2}{2n}.$$

3. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\lambda) = \frac{2\sqrt{\lambda}}{\sqrt{\pi}}\, e^{-\lambda x^2}, \qquad x > 0, \qquad \lambda > 0.$$

a) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n} \left(\frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x_i^2} \right).$$

$$\ln L(\lambda) = n \cdot \ln 2 + \frac{n}{2} \cdot \ln \lambda - \frac{n}{2} \cdot \ln \pi - \lambda \cdot \sum_{i=1}^{n} x_i^2 .$$

$$(\ln L(\lambda))' = \frac{n}{2\lambda} - \sum_{i=1}^{n} x_i^2 = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{n}{2 \sum_{i=1}^{n} X_i^2}.$$

b) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find the maximum likelihood estimate of λ.

$x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.

$$\sum_{i=1}^{n} x_i^2 = 4.5. \qquad \hat{\lambda} = \frac{\mathbf{4}}{\mathbf{9}} \approx 0.444.$$

c) Obtain the method of moments estimator of λ, $\tilde{\lambda}$.

$$\mathrm{E}(\mathrm{X}) = \int_0^\infty x \cdot \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x^2} dx \qquad u = \lambda x^2 \qquad du = 2\lambda x\, dx$$

$$= \int_0^\infty \frac{1}{\sqrt{\pi\lambda}} e^{-u} du = \frac{1}{\sqrt{\pi\lambda}}.$$

$$\overline{\mathrm{X}} = \frac{1}{\sqrt{\pi\lambda}}. \qquad \Rightarrow \qquad \tilde{\lambda} = \frac{1}{\pi\left(\overline{\mathrm{X}}\right)^2}.$$

OR

$$\mathrm{E}(\mathrm{X}^2) = \int_0^\infty x^2 \cdot \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x^2} dx = \ldots = \frac{1}{2\lambda}.$$

$$\overline{\mathrm{X}^2} = \frac{1}{n} \cdot \sum_{i=1}^n \mathrm{X}_i^2 = \frac{1}{2\lambda} \qquad \tilde{\lambda} = \frac{1}{2\overline{\mathrm{X}^2}} = \frac{n}{2\sum_{i=1}^n \mathrm{X}_i^2}$$

d) Suppose $n = 4$, and $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.
Find a method of moments estimate of λ.

$x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 1.1$, $x_4 = 1.7$.

$\bar{x} = 0.9$. $\qquad \tilde{\lambda} \approx 0.392975$.

OR $\qquad \sum_{i=1}^n x_i^2 = 4.5$. $\qquad \tilde{\lambda} = \frac{4}{9} \approx 0.444444$.

e) Find a closed-form expression for $\mathrm{E}(\mathrm{X}^k)$, $k > -1$.

$$\mathrm{E}(\mathrm{X}^k) = \int_0^\infty x^k \cdot \frac{2\sqrt{\lambda}}{\sqrt{\pi}} e^{-\lambda x^2}\, dx \qquad\qquad u = \lambda x^2 \qquad\qquad du = 2\lambda x\, dx$$

$$= \int_0^\infty \left(\frac{u}{\lambda}\right)^{(k-1)/2} \frac{1}{\sqrt{\pi\lambda}} e^{-u}\, du = \frac{1}{\lambda^{k/2}} \frac{1}{\sqrt{\pi}} \int_0^\infty u^{\frac{k+1}{2}-1} e^{-u}\, du$$

$$= \frac{1}{\lambda^{k/2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right).$$

For example,

$$\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^1) = \frac{1}{\lambda^{1/2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1+1}{2}\right) = \frac{1}{\lambda^{1/2}} \frac{1}{\sqrt{\pi}} \Gamma(1) = \frac{1}{\sqrt{\pi\lambda}}.$$

$$\mathrm{E}(\mathrm{X}^2) = \frac{1}{\lambda^{2/2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{2+1}{2}\right) = \frac{1}{\lambda} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{\lambda} \frac{1}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{\lambda} \frac{1}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} = \frac{1}{2\lambda}.$$

$$\mathrm{E}(\mathrm{X}^3) = \frac{1}{\lambda^{3/2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{3+1}{2}\right) = \frac{1}{\lambda^{3/2}} \frac{1}{\sqrt{\pi}} \Gamma(2) = \frac{1}{\sqrt{\pi}\,\lambda^{3/2}}.$$

$$\mathrm{E}(\mathrm{X}^4) = \frac{1}{\lambda^{4/2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{4+1}{2}\right) = \frac{1}{\lambda^2} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{5}{2}\right) = \frac{1}{\lambda^2} \frac{1}{\sqrt{\pi}} \frac{3}{2} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{\lambda^2} \frac{1}{\sqrt{\pi}} \frac{3}{2} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\lambda^2} \frac{1}{\sqrt{\pi}} \frac{3}{4} \sqrt{\pi} = \frac{3}{4\lambda^2}.$$

$$\mathrm{E}(\mathrm{X}^5) = \frac{1}{\lambda^{5/2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{5+1}{2}\right) = \frac{1}{\lambda^{5/2}} \frac{1}{\sqrt{\pi}} \Gamma(3) = \frac{2}{\sqrt{\pi}\,\lambda^{5/2}}.$$

4. Let $\lambda > 0$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\lambda) = \frac{\lambda^2}{2} e^{-\lambda\sqrt{x}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

a) Find the maximum likelihood estimator of λ, $\hat{\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n} \left(\frac{\lambda^2}{2} e^{-\lambda\sqrt{x_i}} \right) = \frac{\lambda^{2n}}{2^n} e^{-\lambda\Sigma\sqrt{x_i}}.$$

$$\ln L(\lambda) = 2n \cdot \ln\lambda - n \cdot \ln 2 - \lambda \cdot \sum_{i=1}^{n} \sqrt{x_i}.$$

$$(\ln L(\lambda))' = \frac{2n}{\lambda} - \sum_{i=1}^{n} \sqrt{x_i} = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{2n}{\sum_{i=1}^{n} \sqrt{X_i}}.$$

b) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$.
Find the maximum likelihood estimate of λ, $\hat{\lambda}$.

$$\sum_{i=1}^{n} \sqrt{x_i} = 0.9 + 1.4 + 0.6 + 0.3 = 3.2.$$

$$\hat{\lambda} = \frac{2 \cdot 4}{3.2} = \mathbf{2.5}.$$

c) Find a closed-form expression for $E(X^k)$ for $k > -1$.

"Hint" 1: $u = \sqrt{x}$.

"Hint" 2: $\frac{\lambda^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\lambda u}$ is the p.d.f. of Gamma$(\alpha,\ \theta = \frac{1}{\lambda})$ distribution

$$E(X^k) = \int_0^\infty x^k \cdot \frac{\lambda^2}{2} e^{-\lambda\sqrt{x}}\, dx \qquad u = \sqrt{x} \qquad x = u^2 \qquad dx = 2u\, du$$

$$= \int_0^\infty u^{2k} \frac{\lambda^2}{2} e^{-\lambda u}\, 2u\, du$$

$$= \frac{\Gamma(2k+2)}{\lambda^{2k}} \int_0^\infty \frac{\lambda^{2k+2}}{\Gamma(2k+2)} u^{(2k+2)-1} e^{-\lambda u}\, du = \frac{\Gamma(2k+2)}{\lambda^{2k}}.$$

d) Find $E(X)$ and $Var(X)$.

$$E(X) = E(X^1) = \frac{\Gamma(4)}{\lambda^2} = \frac{3!}{\lambda^2} = \frac{6}{\lambda^2}.$$

$$E(X^2) = \frac{\Gamma(6)}{\lambda^4} = \frac{5!}{\lambda^4} = \frac{120}{\lambda^4}.$$

$$Var(X) = \frac{120}{\lambda^4} - \left(\frac{6}{\lambda^2}\right)^2 = \frac{84}{\lambda^4}.$$

e) Find a method of moments estimator of λ, $\tilde{\lambda}$.

$$E(X) = \frac{6}{\lambda^2}.$$

$$\overline{X} = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i = \frac{6}{\tilde{\lambda}^2}.$$

$$\Rightarrow \quad \tilde{\lambda} = \sqrt{\frac{6}{\overline{X}}} = \sqrt{\frac{6 \cdot n}{\sum_{i=1}^{n} X_i}}.$$

f) Suppose $n = 4$, and $x_1 = 0.81$, $x_2 = 1.96$, $x_3 = 0.36$, $x_4 = 0.09$.

Find a method of moments estimate of λ, $\tilde{\lambda}$.

$$\sum_{i=1}^{n} x_i = 3.22. \qquad \overline{x} = 0.805.$$

$$\tilde{\lambda} = \sqrt{\frac{6}{0.805}} \approx \mathbf{2.73}.$$

5. Let $\lambda > 0$ and let $X_1, X_2, \dots, X_n$ be independent random variables, each with the probability density function

$$f(x; \lambda) = \frac{\lambda}{x^{\lambda+1}}, \qquad x > 1.$$

a) (i) Find the maximum likelihood estimator of λ, $\hat{\lambda}$.

(ii) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$.

Find the maximum likelihood estimate of λ.

(i) $$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda}{x_i^{\lambda+1}} = \frac{\lambda^n}{\left(\prod_{i=1}^{n} x_i\right)^{\lambda+1}}.$$

$$\ln L(\lambda) = n \ln \lambda - (\lambda + 1) \cdot \sum_{i=1}^{n} \ln x_i .$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \ln x_i = 0. \qquad \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \ln X_i}.$$

(ii) $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$. $\quad \sum_{i=1}^{n} \ln x_i = \ln 54.6 \approx 4.$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} \ln x_i} = \mathbf{\frac{5}{4}} = \mathbf{1.25}.$$

b) (i) Find a method of moments estimator of λ, $\tilde{\lambda}$. (Assume $\lambda > 1$.)

(ii) Suppose $n = 5$, and $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$.

Find a method of moments estimate of λ.

(i) $$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_{1}^{\infty} x \cdot \frac{\lambda}{x^{\lambda+1}}\,dx = \frac{\lambda}{\lambda - 1}.$$

$$\frac{1}{n} \cdot \sum_{i=1}^{n} X_i = \overline{X} = \frac{\tilde{\lambda}}{\tilde{\lambda} - 1}. \qquad \Rightarrow \qquad \tilde{\lambda} = \frac{\overline{X}}{\overline{X} - 1}.$$

(ii) $x_1 = 1.3$, $x_2 = 1.4$, $x_3 = 2$, $x_4 = 3$, $x_5 = 5$. $\sum_{i=1}^{n} x_i = 12.7$.

$$\bar{x} = 2.54. \qquad \tilde{\lambda} = \frac{2.54}{2.54 - 1} \approx \mathbf{1.64935}.$$

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability mass function

$$P(X_i = 1) = \frac{\theta}{3+\theta}, \quad P(X_i = 2) = \frac{2}{3+\theta}, \quad P(X_i = 3) = \frac{1}{3+\theta}, \qquad \theta > 0.$$

a) Obtain the method of moments estimator $\tilde{\theta}$ of θ.

$$E(X) = 1 \times \frac{\theta}{3+\theta} + 2 \times \frac{2}{3+\theta} + 3 \times \frac{1}{3+\theta} = \frac{\theta+7}{3+\theta}.$$

$$\frac{1}{n} \cdot \sum_{i=1}^{n} x_i = \bar{x} = \frac{\tilde{\theta}+7}{3+\tilde{\theta}}. \qquad\qquad 3\bar{x} + \tilde{\theta}\,\bar{x} = \tilde{\theta} + 7.$$

$$\Rightarrow \qquad \tilde{\theta} = \frac{7-3\bar{x}}{\bar{x}-1}.$$

b) Obtain the maximum likelihood estimator $\hat{\theta}$ of θ.

$$L(\theta) = \frac{1}{(3+\theta)^n} \cdot \theta^{(\#\text{ of 1's})} \cdot 2^{(\#\text{ of 2's})} \cdot 1^{(\#\text{ of 3's})}.$$

$$\ln L(\theta) = -n\ln(3+\theta) + (\#\text{ of 1's})\ln(\theta) + (\#\text{ of 2's})\ln(2) + (\#\text{ of 3's})\ln(1).$$

$$(\ln L(\theta))' = -\frac{n}{3+\theta} + \frac{(\#\text{ of 1's})}{\theta} = 0 \qquad\qquad \Rightarrow \qquad \hat{\theta} = \frac{3 \cdot (\#\text{ of 1's})}{n - (\#\text{ of 1's})}.$$

7. Bert and Ernie find a coin on the sidewalk on Sesame Street. They wish to estimate p, the probability of Heads. Bert got X Heads in N coin tosses (N is fixed, X is random). Ernie got Heads for the first time on the Y^{th} coin toss (Y is random). They decide to combine their information in hope of a better estimate. (Assume independence.)

a) What is the likelihood function $\text{L}(p) = \text{L}(p;\, \text{X}, \text{N}, \text{Y})$?

X has a Binomial(N, p) distribution. Y has a Geometric(p) distribution.

$$\text{L}(p) = \binom{\text{N}}{\text{X}} p^{\text{X}} (1-p)^{\text{N}-\text{X}} \times (1-p)^{\text{Y}-1}\, p = \binom{\text{N}}{\text{X}} p^{\text{X}+1} (1-p)^{\text{N}-\text{X}+\text{Y}-1}.$$

b) Obtain the maximum likelihood estimator for p.

$$\ln \text{L}(p) = \ln \binom{\text{N}}{\text{X}} + (\text{X}+1)\ln p + (\text{N}-\text{X}+\text{Y}-1)\ln(1-p).$$

$$\frac{d}{dp} \ln \text{L}(p) = \frac{\text{X}+1}{p} - \frac{\text{N}-\text{X}+\text{Y}-1}{1-p} = \frac{\text{X}+1-\text{X}p-p-\text{N}p+\text{X}p-\text{Y}p+p}{p(1-p)}$$

$$= \frac{\text{X}+1-\text{N}p-\text{Y}p}{p(1-p)} = 0.$$

$$\Rightarrow \qquad \hat{p} = \frac{\text{X}+1}{\text{N}+\text{Y}}.$$

c) Explain intuitively why your estimator makes good sense.

Bert:	N attempts,	X "successes"
Ernie:	Y attempts,	1 "success"

$$\hat{p} = \frac{\text{X}+1}{\text{N}+\text{Y}} = \frac{\text{total number of "successes"}}{\text{total number of attempts}}.$$

8. Let $\theta \in \mathbb{R}$ and let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}, \qquad x \in \mathbb{R}.$$

a) Find a method of moments estimator $\tilde{\theta}$ of θ.

$f(x;\theta)$ is symmetric about θ.

$\Rightarrow$ $\quad \mathrm{E}(X) = \theta \quad$ (balancing point) $\qquad \tilde{\theta} = \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$.

b) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

$$L(\theta) = \frac{1}{2^n} \exp\left\{ -\sum_{i=1}^{n} |x_i - \theta| \right\}.$$

$\Rightarrow$ $\quad$ To maximize $L(\theta)$, we need to minimize $\sum_{i=1}^{n} |x_i - \theta|$.

Let y_k denote the k^{th} smallest among $x_1, x_2, \ldots, x_n$.
($y_1 = \min x_i,\ y_n = \max x_i$.)

If $\theta \in (y_k, y_{k+1})$, $\qquad \frac{d}{d\theta} \sum_{i=1}^{n} |x_i - \theta| = k - (n-k) = 2k - n,$

$$\frac{d}{d\theta} \sum_{i=1}^{n} |x_i - \theta| < 0 \text{ if } k < \frac{n}{2}, \qquad \frac{d}{d\theta} \sum_{i=1}^{n} |x_i - \theta| > 0 \text{ if } k > \frac{n}{2}.$$

If n is odd, $\hat{\theta} = Y_{\frac{n+1}{2}}$ (the middle value in the data set).

If n is even, $\hat{\theta} \in [Y_{\frac{n}{2}}, Y_{\frac{n}{2}+1}]$ (any value between the middle two).

For example, $\hat{\theta}$ = sample median.

9. A random sample of size $n = 16$ from $N(\mu, \sigma^2 = 64)$ yielded $\overline{x} = 85$.
Construct the following confidence intervals for μ:

$\overline{x} = 85$ $\sigma = 8$ $n = 16$

σ is known. The confidence interval : $\overline{X} \pm z_{\alpha/2} \dfrac{\sigma}{\sqrt{n}}$.

a) 95%.

$\alpha = 0.05$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = 1.96.$

$85 \pm 1.96 \cdot \dfrac{8}{\sqrt{16}}$ **85 ± 3.92** **(81.08 ; 88.92)**

b) 90%.

$\alpha = 0.10$ $\alpha/2 = 0.05.$ $z_{\alpha/2} = 1.645.$

$85 \pm 1.645 \cdot \dfrac{8}{\sqrt{16}}$ **85 ± 3.29** **(81.71 ; 88.29)**

c) 80%.

$\alpha = 0.20$ $\alpha/2 = 0.10.$ $z_{\alpha/2} = 1.28.$

$85 \pm 1.28 \cdot \dfrac{8}{\sqrt{16}}$ **85 ± 2.56** **(82.44 ; 87.56)**

OR

$\alpha = 0.20$ $\alpha/2 = 0.10.$ $z_{\alpha/2} = 1.282.$

$85 \pm 1.282 \cdot \dfrac{8}{\sqrt{16}}$ **85 ± 2.564** **(82.436 ; 87.564)**

10. What is the minimum sample size required for estimating μ for $N(\mu, \sigma^2 = 64)$ to within ± 3 with confidence level

$\varepsilon = 10$, $\sigma = 8$.

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{z_{\alpha/2} \cdot 8}{3} \right)^2 .$$

a) 95%. $\alpha = 0.05$ $\alpha/2 = 0.025$. $z_{\alpha/2} = 1.96$.

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{1.96 \cdot 8}{3} \right)^2 \approx 27.318.$$

Round **up**. $n = \mathbf{28}$.

b) 90%. $\alpha = 0.10$ $\alpha/2 = 0.05$. $z_{\alpha/2} = 1.645$.

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{1.645 \cdot 8}{3} \right)^2 \approx 19.243.$$

Round **up**. $n = \mathbf{20}$.

c) 80%. $\alpha = 0.20$ $\alpha/2 = 0.10$. $z_{\alpha/2} = 1.28$.

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{1.28 \cdot 8}{3} \right)^2 \approx 11.651.$$

Round **up**. $n = \mathbf{12}$.

OR

$\alpha = 0.20$ $\alpha/2 = 0.10$. $z_{\alpha/2} = 1.282$.

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{1.282 \cdot 8}{3} \right)^2 \approx 11.687.$$

Round **up**. $n = \mathbf{12}$.

11. Suppose the overall (population) standard deviation of the bill amounts at a supermarket is $\sigma = \$13.75$.

a) Find the probability that the sample mean bill amount will be within \$2.00 of the overall mean bill amount for a random sample of 121 customers.

Need $P(\mu - 2.00 \le \overline{X} \le \mu + 2.00) = ?$

$n = 121$ – large Central Limit Theorem: $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} = Z.$

$$P(\mu - 2.00 \le \overline{X} \le \mu + 2.00) = P\left(\frac{(\mu - 2.00) - \mu}{13.75 / \sqrt{121}} \le Z \le \frac{(\mu + 2.00) - \mu}{13.75 / \sqrt{121}} \right)$$

$$= P(-1.60 \le Z \le 1.60) = 0.9452 - 0.0548 = \mathbf{0.8904}.$$

b) What is the minimum sample size required for estimating the overall mean bill amount to within \$2.00 with 95% confidence?

$\varepsilon = 2.00,$ $\sigma = 13.75,$ $\alpha = 0.05,$ $\alpha/2 = 0.025,$ $z_{\alpha/2} = z_{0.035} = 1.96.$

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{\varepsilon} \right)^2 = \left(\frac{1.96 \cdot 13.75}{2.00} \right)^2 = 181.575625.$$

Round **up**. $n = \mathbf{182}$.

12. 11. (continued)

The supermarket selected a random sample of 121 customers, which showed the sample mean bill amount of \$78.80.

$\overline{X} = \$78.80,$ $\sigma = \$13.75,$ $n = 121.$

c) Construct a 95% confidence interval for the overall mean bill amount at this supermarket.

σ is known. $n = 121$ – large.

The confidence interval for μ: $\overline{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$.

$\alpha = 0.05.$ $\alpha/2 = 0.025.$ $z_{\alpha/2} = z_{0.025} = 1.96.$

$78.80 \pm 1.96 \cdot \frac{13.75}{\sqrt{121}}$ $\mathbf{78.80 \pm 2.45}$ $\mathbf{(76.35\ ;\ 81.25)}$

d) Suppose the supermarket puts Alex in charge of computing the confidence interval, and he gets the answer $(76.15, 81.45)$. Alex says that he used a different confidence level, but other than that did everything correctly. Find the confidence level used by Alex.

$\overline{X} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$ $81.45 - 78.80 = 78.80 - 76.15 = 2.65.$

$2.65 = z_{\alpha/2} \cdot \frac{13.75}{\sqrt{121}}$ $z_{\alpha/2} = 2.12.$

$\alpha/2$ = Area to the right of 2.12 = 0.0170. $\alpha = 2 \cdot 0.0170 = 0.0340.$

Confidence level = $100 \cdot (1 - \alpha)\%$ = **96.6%**.