STAT 400
UIUC

Examples for 1.2

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Multiplication Principle (Fundamental Rule of Counting):

If there are n events and event i can occur in N_i possible ways, then the number of ways in which the sequence of n events may occur is

$$N_1 \cdot N_2 \cdot \ldots \cdot N_n$$

1. Manager of a radio station decided that every day the broadcast will start with one of the 9 Beethoven Symphonies, followed by one of Mozart's 27 Piano Concertos, followed by one of Schubert’s 15 String Quartets. Approximately how many years can the station do that without repeating the program?

$$\underbrace{9}_{\text{Beethoven Symphony}} \bullet \underbrace{27}_{\text{Mozart's Piano Concerto}} \bullet \underbrace{15}_{\text{Schubert's String Quartet}} = \mathbf{3645}\text{ days}$$

3645 days $\approx$ **10** years.

1½. The call letters of radio and television stations in the United States begin with either K or W. Those west of the Mississippi River start with K and those east of it with W.

a) Some stations, such as KID in Idaho Falls, Idaho, and WOW in Omaha, Nebraska, have 3 call letters. How many sets of call letters having 3 letters re possible?

$2 \times 26 \times 26 =$ **1,352**.

b) Most stations that were licensed after 1927 have 4 call letters, such as KUZZ in Bakersfield, California, and WXYZ in Detroit, Michigan. How many sets of call letters having 4 letters are possible?

$2 \times 26 \times 26 \times 26 =$ **35,152**.

c) How many sets of call letters having 4 letters are possible if we are not allowed to repeat letters?

$2 \times 25 \times 24 \times 23 =$ **27,600**.

2. In how many orders can the names of 5 candidates for the same office be listed on a ballot?

$$\underset{1st}{5} \bullet \underset{2nd}{4} \bullet \underset{3rd}{3} \bullet \underset{4th}{2} \bullet \underset{5th}{1} = \mathbf{120}.$$

$$n\,! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$$

$$n\,! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 \qquad\qquad 0\,! = 1$$

$n\,!$ is the number of ways to rearrange (reorder) n distinct items.

For example, there are $7\,! = 5{,}040$ different ways to arrange 7 books on a bookshelf.

3. How many ways are there of scrambling the letters of the word SCRAMBLE ?

There are 8 letters in the word SCRAMBLE, none of them repeating.
Therefore, there are $8\,!$ = **40,320** different ways to rearrange the letters.

4. Eight horses are entered in a race in which bets are placed on which horse will win, place, and show (that is, finish first, second and third). Suppose that the race is run and there are no ties.

a) In how many orders can all eight horses finish the race?

$8\,!$ = **40,320** different ways for eight horses to finish the race.

b) In how many ways can the win, place, and show be taken?

$$\underset{1st}{8} \bullet \underset{2nd}{7} \bullet \underset{3rd}{6} = \mathbf{336}.$$

$$8 \bullet 7 \bullet 6 = \frac{8 \bullet 7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}{5 \bullet 4 \bullet 3 \bullet 2 \bullet 1} = \frac{8\,!}{5\,!}$$

Permutations are the possible ordered selections of r objects out of a total of n objects. The number of permutations of n objects taken r at a time is

$${}_n\mathrm{P}_r = \frac{n!}{(n-r)!}$$

For example, there are ${}_{11}P_4 = 11 \bullet 10 \bullet 9 \bullet 8 = 7920$ different ways to appoint a president, a vice-president, a secretary, and a treasurer for a club that has 11 members.

5. At *Momma Leona's Pizza* you can get a pizza with or without each of eight different toppings. How many different three-topping pizzas can you get at *Momma Leona's Pizza* if each topping can be put on a pizza at most once? (The order in which the toppings toppings are selected does not matter.)

There are ${}_8P_3 = 336$ ordered selections of three toppings out of eight.

Each unordered selection is repeated $6 = 3!$ times

(ABC, ACB, BAC, BCA, CAB, CBA).

Therefore, the number of unordered selections of three toppings out of eight is

$$\frac{{}_8P_3}{3!} = \frac{8!}{3!\cdot(8-3)!} = \frac{336}{6} = \mathbf{56}.$$

Combinations are the possible selections of r items from a group of n items *regardless of the order of selection*. The number of combinations of n objects taken r at a time is

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$$

Pascal Triangle:

1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5
1 6 15 20 15 6 1	6
1 7 21 35 35 21 7 1	7
1 8 28 56 70 56 28 8 1	8
1 9 36 84 126 126 84 36 9 1	9
1 10 45 120 210 252 210 120 45 10 1	10

For example, ${}_8C_0 = 1$, ${}_8C_1 = 8$, ${}_8C_2 = 28$, ${}_8C_3 = 56$, ${}_8C_4 = 70$,
${}_8C_5 = 56$, ${}_8C_6 = 28$, ${}_8C_7 = 8$, ${}_8C_8 = 1$.

$$\binom{n}{r} = \binom{n}{n-r}. \qquad \binom{n}{0} = \binom{n}{n} = 1. \qquad \binom{n}{1} = \binom{n}{n-1} = n. \qquad \binom{n}{2} = \binom{n}{n-2} = \frac{n \cdot (n-1)}{2}.$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \, y^{n-k}. \qquad \sum_{k=0}^{n} \binom{n}{k} = 2^n. \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

	order of the selection is important	order of the selection is **not** important
repetitions allowed (w/ replacement)	n^r	${}_{n+r-1}\mathrm{C}_r$
repetitions **not** allowed (w/o replacement)	${}_n\mathrm{P}_r = \dfrac{n!}{(n-r)!}$	${}_n\mathrm{C}_r = \dfrac{n!}{r! \cdot (n-r)!}$

6. The Baskin-Robbins Ice Cream Stores have 31 flavors of ice cream.

a) How many different 3-scoop ice cream cones are possible if you are allowed to repeat flavors and want the scoops put on the cone in a particular order?

w/ replacement order of the selection is important

$31^3 =$ **29,791**. **OR** $\underset{1st}{31} \cdot \underset{2nd}{31} \cdot \underset{3rd}{31} =$ **29,791**.

b) How many different 3-scoop ice cream cones are possible if each scoop is a different flavor and you want the scoops put on the cone in a particular order?

w/o replacement order of the selection is important

${}_{31}P_3 = \dfrac{31!}{28!} =$ **26,970**. **OR** $\underset{1st}{31} \cdot \underset{2nd}{30} \cdot \underset{3rd}{29} =$ **26,970**.

c) How many different 3-scoop cones are possible if each scoop is a different flavor and you don't care about their order on the cone?

w/o replacement order of the selection is not important

${}_{31}C_3 = \dbinom{31}{3} = \dfrac{31!}{3! \cdot 28!} =$ **4,495**.

d) How many different 3-scoop ice cream cones are possible if you are allowed to repeat flavors, but the order in which the scoops are placed into the cone is not important?

w/ replacement order of the selection is not important

$$_{31+3-1}C_3 = \binom{33}{3} = \frac{33!}{3! \cdot 30!} = \mathbf{5{,}456}.$$

7. To play Michigan Lotto, a person must pick 6 numbers from 49 numbers.

a) If the player matches all 6 numbers (6 of 6) drawn, he/she wins the grand prize jackpot. Find the probability of winning the jackpot.

Total number of possible outcomes = $_{49}C_6$ = **13,983,816**.

$$\text{P(Jackpot)} = \frac{\mathbf{1}}{\mathbf{13{,}983{,}816}} \approx \mathbf{0.0000000715}.$$

b) Find the probability of guessing correctly 4 out of 6 numbers.

Guessing correctly 4 out of 6 numbers = **two** stages:

(1) guessing 4 numbers correctly (out of 6) $_6C_4$ = **15**.

(2) guessing 2 numbers incorrectly (out of 43) $_{43}C_2$ = **903**.

$$\begin{array}{ccc} & 49 & \\ \swarrow & & \searrow \\ 6 & & 43 \\ \text{good} & & \text{bad} \\ \downarrow & & \downarrow \\ 4 & & 2 \end{array}$$

$$\frac{_6C_4 \cdot {}_{43}C_2}{_{49}C_6} = \frac{15 \cdot 903}{13{,}983{,}816}$$

$$= \frac{13{,}545}{13{,}983{,}816} \approx \mathbf{0.00096862}.$$

Back to part (a):

$$\begin{array}{ccc} & 49 & \\ \swarrow & & \searrow \\ 6 & & 43 \\ \text{good} & & \text{bad} \\ \downarrow & & \downarrow \\ 6 & & 0 \end{array}$$

$$\frac{_6C_6 \cdot {}_{43}C_0}{_{49}C_6} = \frac{1 \cdot 1}{13{,}983{,}816}$$

$$= \frac{1}{13{,}983{,}816} \approx \mathbf{0.0000000715}.$$

8. How many ways are there of scrambling the letters of the word BANANA ?

6 letters can be rearranged in $6\,! = 720$ different ways.

However, there are $3\,! = 6$ different ways to rearrange repeating A's,

$2\,! = 2$ different ways to rearrange repeating N's,

$1\,! = 1$ ways to "rearrange" B.

Therefore, there are $\dfrac{6!}{3! \cdot 2! \cdot 1!} = \mathbf{60}$ different ways to rearrange the letters.

9. A student bookstore has 18 STAT 400 textbooks: 9 are new, 6 are used, and 3 are abused.

a) How many ways are there to arrange the textbooks on the shelf?

$$\frac{18!}{9! \cdot 6! \cdot 3!} = \mathbf{4{,}084{,}080} \qquad \text{OR} \qquad \binom{18}{9} \cdot \binom{9}{6} \cdot \binom{3}{3} = 48620 \cdot 84 \cdot 1 = \mathbf{4{,}084{,}080}.$$

b) Eight students come to the store to buy a STAT 400 textbook. Suppose that the purchased textbooks are selected at random. What is the probability that 4 of the students would get a new book, 3 would get a used one, and 1 student would get an abused textbook?

$$\begin{array}{ccc} & 18 & \\ \swarrow & \downarrow & \searrow \\ 9 & 6 & 3 \\ \text{new} & \text{used} & \text{abused} \\ \downarrow & \downarrow & \downarrow \\ 4 & 3 & 1 \end{array}$$

$$\frac{\binom{9}{4} \cdot \binom{6}{3} \cdot \binom{3}{1}}{\binom{18}{8}} = \frac{126 \cdot 20 \cdot 3}{43{,}758}$$

$$= \frac{7{,}560}{43{,}758} \approx \mathbf{0.1728}.$$

10. What is the probability that a 5-card poker hand drawn from a standard 52-card deck has …

a) 2 red and 3 black cards?

$$\begin{array}{ccc} & 52 & \\ \swarrow & & \searrow \\ 26 & & 26 \\ \text{red} & & \text{black} \\ \downarrow & & \downarrow \\ 2 & & 3 \end{array}$$

$$\frac{{}_{26}C_2 \cdot {}_{26}C_3}{{}_{52}C_5} = \frac{325 \cdot 2{,}600}{2{,}598{,}960}$$

$$= \frac{845{,}000}{2{,}598{,}960} \approx \mathbf{0.32513}.$$

b) 2 clubs, 2 spades and 1 diamond?

$$\begin{array}{cccc} & & 52 & \\ \swarrow & \swarrow & \searrow & \searrow \\ 13 & 13 & 13 & 13 \\ \text{clubs} & \text{spades} & \text{diamonds} & \text{hearts} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 2 & 1 & 0 \end{array}$$

$$\frac{{}_{13}C_2 \cdot {}_{13}C_2 \cdot {}_{13}C_1 \cdot {}_{13}C_0}{{}_{52}C_5} = \frac{78 \cdot 78 \cdot 13 \cdot 1}{2{,}598{,}960} = \frac{79{,}092}{2{,}598{,}960} \approx \mathbf{0.03}.$$

c) exactly 3 queens?

$$\begin{array}{ccc} & 52 & \\ \swarrow & & \searrow \\ 4 & & 48 \\ \text{queens} & & \text{other} \\ \downarrow & & \downarrow \\ 3 & & 2 \end{array}$$

$$\frac{{}_{4}C_3 \cdot {}_{48}C_2}{{}_{52}C_5} = \frac{4 \cdot 1{,}128}{2{,}598{,}960}$$

$$= \frac{4{,}512}{2{,}598{,}960} \approx \mathbf{0.001736}.$$

d) 1 ace and 2 face cards (king, queen, jack)?

$$\begin{array}{ccc} & 52 & \\ \swarrow & \downarrow & \searrow \\ 4 & 12 & 36 \\ \text{aces} & \text{face} & \text{other} \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 2 \end{array}$$

$$\frac{{}_{4}C_1 \cdot {}_{12}C_2 \cdot {}_{36}C_2}{{}_{52}C_5} = \frac{4 \cdot 66 \cdot 630}{2{,}598{,}960}$$

$$= \frac{166{,}320}{2{,}598{,}960} \approx \mathbf{0.064}.$$

e) no aces?

$$\begin{array}{ccc} & 52 & \\ \swarrow & & \searrow \\ 4 & & 48 \\ \text{aces} & & \text{other} \\ \downarrow & & \downarrow \\ 0 & & 5 \end{array}$$

$$\frac{{}_4C_0 \cdot {}_{48}C_5}{{}_{52}C_5} = \frac{1 \cdot 1{,}712{,}304}{2{,}598{,}960}$$

$$= \frac{1{,}712{,}304}{2{,}598{,}960} \approx \mathbf{0.658842}.$$

OR

$$\frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \cdot \frac{44}{48} \approx \mathbf{0.658842}.$$

f) at least one ace?

P(at least one ace) = 1 – P(no aces) ≈ **0.341158**.

OR

P(at least one ace) = P(1 ace) + P(2 aces) + P(3 aces) + P(4 aces)

$$= \frac{{}_4C_1 \cdot {}_{48}C_4}{{}_{52}C_5} + \frac{{}_4C_2 \cdot {}_{48}C_3}{{}_{52}C_5} + \frac{{}_4C_3 \cdot {}_{48}C_2}{{}_{52}C_5} + \frac{{}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5}$$

$$= \frac{4 \cdot 194{,}580}{2{,}598{,}960} + \frac{6 \cdot 17{,}296}{2{,}598{,}960} + \frac{4 \cdot 1{,}128}{2{,}598{,}960} + \frac{1 \cdot 48}{2{,}598{,}960} \approx \mathbf{0.341158}.$$

g) two aces or two kings, or both (two aces and two kings)?

$$P(\,2A \cup 2K\,) = P(\,2A\,) + P(\,2K\,) - P(\,2A \cap 2K\,)$$

$$= \frac{{}_4C_2 \cdot {}_{48}C_3}{{}_{52}C_5} + \frac{{}_4C_2 \cdot {}_{48}C_3}{{}_{52}C_5} - \frac{{}_4C_2 \cdot {}_4C_2 \cdot {}_{44}C_1}{{}_{52}C_5}$$

$$= 0.03993 + 0.03993 - 0.00061 = \mathbf{0.07925}.$$

h) at least 2 hearts, given that there are at most 4 hearts?

$$P(\text{ at least 2 hearts} \mid \text{at most 4 hearts }) = \frac{P(\text{ at least 2 hearts} \cap \text{at most 4 hearts })}{P(\text{ at most 4 hearts })}$$

$$= \frac{P(\text{ 2 hearts }) + P(\text{ 3 hearts }) + P(\text{ 4 hearts })}{P(\text{ 0 hearts }) + P(\text{ 1 heart }) + P(\text{ 2 hearts }) + P(\text{ 3 hearts }) + P(\text{ 4 hearts })}$$

$$= \frac{\dfrac{{}_{13}C_2 \cdot {}_{39}C_3}{{}_{52}C_5} + \dfrac{{}_{13}C_3 \cdot {}_{39}C_2}{{}_{52}C_5} + \dfrac{{}_{13}C_4 \cdot {}_{39}C_1}{{}_{52}C_5}}{\dfrac{{}_{13}C_0 \cdot {}_{39}C_5}{{}_{52}C_5} + \dfrac{{}_{13}C_1 \cdot {}_{39}C_4}{{}_{52}C_5} + \dfrac{{}_{13}C_2 \cdot {}_{39}C_3}{{}_{52}C_5} + \dfrac{{}_{13}C_3 \cdot {}_{39}C_2}{{}_{52}C_5} + \dfrac{{}_{13}C_4 \cdot {}_{39}C_1}{{}_{52}C_5}}$$

$$= \frac{{}_{13}C_2 \cdot {}_{39}C_3 + {}_{13}C_3 \cdot {}_{39}C_2 + {}_{13}C_4 \cdot {}_{39}C_1}{{}_{13}C_0 \cdot {}_{39}C_5 + {}_{13}C_1 \cdot {}_{39}C_4 + {}_{13}C_2 \cdot {}_{39}C_3 + {}_{13}C_3 \cdot {}_{39}C_2 + {}_{13}C_4 \cdot {}_{39}C_1}$$

OR

$$P(\text{ at least 2 hearts } | \text{ at most 4 hearts }) = \frac{P(\text{ at least 2 hearts} \cap \text{at most 4 hearts})}{P(\text{ at most 4 hearts})}$$

$$= \frac{P(\text{ 2 hearts}) + P(\text{ 3 hearts}) + P(\text{ 4 hearts})}{1 - P(\text{ 5 hearts})}$$

$$= \frac{\dfrac{{}_{13}C_2 \cdot {}_{39}C_3}{{}_{52}C_5} + \dfrac{{}_{13}C_3 \cdot {}_{39}C_2}{{}_{52}C_5} + \dfrac{{}_{13}C_4 \cdot {}_{39}C_1}{{}_{52}C_5}}{1 - \dfrac{{}_{13}C_5 \cdot {}_{39}C_0}{{}_{52}C_5}}$$

$$= \frac{{}_{13}C_2 \cdot {}_{39}C_3 + {}_{13}C_3 \cdot {}_{39}C_2 + {}_{13}C_4 \cdot {}_{39}C_1}{{}_{52}C_5 - {}_{13}C_5 \cdot {}_{39}C_0}$$