STAT 400
UIUC

Answers for 1.4

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Events A and B are **independent** if and only if

$$\mathbf{P(B \mid A) = P(B)} \qquad \mathbf{P(A \mid B) = P(A)}$$

$$\mathbf{P(A \cap B) = P(A) \cdot P(B)}$$

Note that if two events, A and B, are mutually exclusive, then $P(A \cap B) = 0$. Therefore, two mutually exclusive events cannot be independent, unless at least one of them has probability 0.

1. The probability that a randomly selected student at Anytown College owns a bicycle is 0.55, the probability that a student owns a car is 0.30, and the probability that a student owns both is 0.10. Are events {a student owns a bicycle} and {a student owns a car} independent?

$P(B \cap C) \neq P(B) \times P(C)$.

$0.10 \neq 0.55 \times 0.30$.

B and C are **NOT independent**.

1½. During the first week of the semester, 80% of customers at a local convenience store bought either beer or potato chips (or both). 60% bought potato chips. 30% of the customers bought both beer and potato chips. Are events {a randomly selected customer bought potato chips} and {a randomly selected customer bought beer} independent?

[Recall that P(Beer) = 0.50.]

$P(B \cap PC) = P(B) \times P(PC)$.

$0.30 = 0.50 \times 0.60$.

B and PC are **independent**.

Events A, B and C are **independent** if and only if

$$P(A \cap B) = P(A) \cdot P(B), \quad P(A \cap C) = P(A) \cdot P(C), \quad P(B \cap C) = P(B) \cdot P(C),$$

$$\text{and} \quad P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

1¾. Suppose that a fair coin is tossed twice. Consider A = {H on the first toss}, B = {H on the second toss} and C = {exactly one H in two tosses}.

S = { TT, TH, HT, HH },

A = { HT, HH }, B = { TH, HH }, C = { TH, HT }.

$P(A) = {}^1/_2$, $P(B) = {}^1/_2$, $P(C) = {}^1/_2$.

a) Are A and B independent?

$A \cap B$ = { HH }, $P(A \cap B) = {}^1/_4$.

$P(A \cap B) = P(A) \times P(B)$. A and B are **independent**.

b) Are A and C independent?

$A \cap C$ = { HT }, $P(A \cap C) = {}^1/_4$.

$P(A \cap C) = P(A) \times P(C)$. A and C are **independent**.

b) Are B and C independent?

$B \cap C$ = { TH }, $P(B \cap C) = {}^1/_4$.

$P(B \cap C) = P(B) \times P(C)$. B and C are **independent**.

d) Are A, B and C independent?

Since $P(A \cap B \cap C) \neq P(A) \times P(B) \times P(C)$, A, B and C are **not** independent, even though A, B and C are pairwise independent.

2. A girl is told by her boyfriend that she is “one in a billion.” She has a dimple in her chin, probability $^1/_{100}$, eyes of different colors, probability $^1/_{1,000}$, and is absolutely crazy about mathematics, probability $^1/_{10,000}$.

a) Do these events seem to be independent or dependent?

Independent. ☺

b) Show why the girl is “one in a billion.”

$^1/_{100} \times {}^1/_{1,000} \times {}^1/_{10,000} = {}^1/_{1,000,000,000} = {}^1/_{1 \text{ billion}}$.

3. Bart and Nelson talked Milhouse into throwing water balloons at Principal Skinner. Suppose that Bart hits his target with probability 0.80, Nelson misses 25% of the time, and Milhouse hits the target half the time. Assume that their attempts are independent of each other.

P(B) = 0.80, P(N) = 0.75, P(M) = 0.50.

8 possible outcomes (not equally likely):

B	N	M
B	N	M′
B	N′	M
B′	N	M
B	N′	M′
B′	N	M′
B′	N′	M
B′	N′	M′

a) Find the probability that all of them will hit Principal Skinner.

P(all) = P(B ∩ N ∩ M) = P(B) × P(N) × P(M) = 0.80 × 0.75 × 0.50 = **0.30**.

b) Find the probability that exactly one of the boys will hit Principal Skinner.

$P(B \cap N' \cap M') = P(B) \times P(N') \times P(M') = 0.80 \times 0.25 \times 0.50 = 0.100$,

$P(B' \cap N \cap M') = P(B') \times P(N) \times P(M') = 0.20 \times 0.75 \times 0.50 = 0.075$,

$P(B' \cap N' \cap M) = P(B') \times P(N') \times P(M) = 0.20 \times 0.25 \times 0.50 = 0.025$.

P(exactly one) = 0.100 + 0.075 + 0.025 = **0.20**.

c) Find the probability that at least one of the boys will hit Principal Skinner.

$$\begin{aligned} P(\text{at least one}) &= 1 - P(\text{none}) = 1 - P(B' \cap N' \cap M') \\ &= 1 - P(B') \times P(N') \times P(M') \\ &= 1 - 0.20 \times 0.25 \times 0.50 = \mathbf{0.975}. \end{aligned}$$

Idea: $P(\text{at least one of } A_i \text{ occurs}) = 1 - P(\text{none of } A_i \text{ occurs})$

$$\mathbf{P(A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_n) = 1 - P((\text{not } A_1) \text{ and } (\text{not } A_2) \text{ and } \ldots \text{ and } (\text{not } A_n))}$$

$$\mathbf{P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(A_1' \cap A_2' \cap \ldots \cap A_n')}$$

For independent events

$$\mathbf{P(A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_n) = 1 - P(\text{not } A_1) \bullet P(\text{not } A_2) \bullet \ldots \bullet P(\text{not } A_n)}$$

$$\mathbf{P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(A_1') \bullet P(A_2') \bullet \ldots \bullet P(A_n')}$$

4. *Often On time Parcel Service* (*OOPS*) delivers a package to the wrong address with probability 0.05 on any delivery. Suppose that each delivery is independent of all the others. There were 7 packages delivered on a particular day. What is the probability that at least one of them was delivered to the wrong address?

P(at least one wrong) = 1 – P(all correct) = $1 - (0.95)^7 \approx$ **0.3016627**.

5. A major oil company has decided to drill independent test wells in the Alaskan wilderness. The probability of any well producing oil is 0.30. Find the probability that the fifth well is the first to produce oil.

$$\begin{array}{ccccccccc} F & & F & & F & & F & & S \\ 0.70 & \cdot & 0.70 & \cdot & 0.70 & \cdot & 0.70 & \cdot & 0.30 \end{array} = \mathbf{0.07203}.$$

5½. Consider a fair (balanced, symmetric) 6-sided die. Assuming that the outcome of each roll is independent of all other rolls, find the probability that the first "6" occurs on the m^{th} roll.

[first (m – 1) rolls] [m^{th} roll]

[no "6" on the 1st roll ∩ … ∩ no "6" on the (m – 1) th roll] ∩ ["6" on the m^{th} roll]

$$\left(\frac{5}{6}\right)^{m-1} \times \left(\frac{1}{6}\right)$$

5¾. Consider a fair (balanced, symmetric) 6-sided die. Assuming that the outcome of each roll is independent of all other rolls, find the probability that "6" occurs before an odd outcome.

$$P(\text{"6" before odd}) = \left(\frac{1}{6}\right) + \left(\frac{2}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{2}{6}\right)^2\left(\frac{1}{6}\right) + \ldots + \left(\frac{2}{6}\right)^k\left(\frac{1}{6}\right) + \ldots = \sum_{k=0}^{\infty}\left(\frac{2}{6}\right)^k\left(\frac{1}{6}\right)$$

$$= \left(\frac{1}{6}\right) \cdot \frac{1}{1-\frac{2}{6}} = \mathbf{\frac{1}{4}} = \mathbf{0.25}.$$

$^2/_6$ = P(not "6" and not odd)

OR

Four "relevant" outcomes: one "good" outcome – "6"
three "bad" outcomes – "1", "3", "5"

P ("6" before odd) = $\mathbf{^1/_4}$ = **0.25**.

6. An automobile salesman thinks that the probability of making a sale is 0.30. If he talks to five customers on a particular day, what is the probability that he will make exactly 2 sales? (Assume independence.)

S S F F F	F S F S F
S F S F F	F S F F S
S F F S F	F F S S F
S F F F S	F F S F S
F S S F F	F F F S S

$\text{P}(\text{exactly 2 sales}) = 10 \cdot (0.30)^2 \cdot (0.70)^3 = \mathbf{0.3087}$.

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6½. A jar has N marbles, K of them are orange and $N - K$ are blue. Suppose marbles are selected from the jar without replacement. Find the probability that the second marble is orange.

$$\text{P}(\text{O}_2) = \text{P}(\text{O}_1\,\text{O}_2) + \text{P}(\text{B}_1\,\text{O}_2) = \frac{K}{N} \cdot \frac{K-1}{N-1} + \frac{N-K}{N} \cdot \frac{K}{N-1}$$

$$= \frac{K}{N(N-1)} \cdot \left[(K-1) + (N-K) \right] = \frac{K}{N}.$$

Note that $\text{P}(\text{O}_2) = \text{P}(\text{O}_1)$.

Similarly, $\text{P}(\text{O}_3) = \text{P}(\text{O}_4) = \ldots = \dfrac{K}{N}$.

However, $\text{P}(\text{O}_1\,\text{O}_2) \neq \text{P}(\text{O}_2) \times \text{P}(\text{O}_1)$.

O_1 and O_2 are NOT independent.

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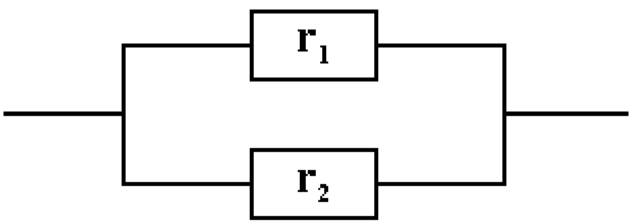
Series Connection:

Reliability:

$$\mathbf{r}_1 \times \mathbf{r}_2$$

In general, $\mathbf{r}_1 \times \mathbf{r}_2 \times \ldots \times \mathbf{r}_k$

Parallel Connection:



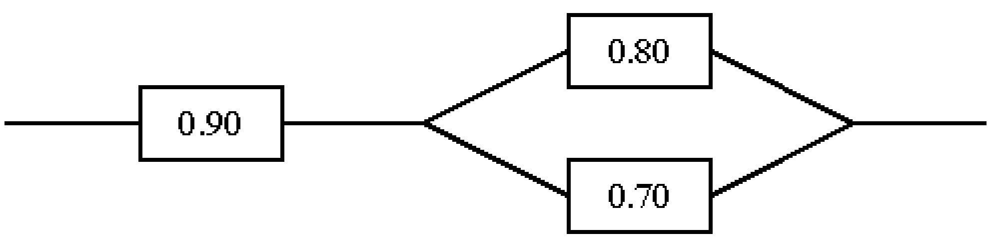
Reliability:

$$1 - (1-\mathbf{r}_1) \times (1-\mathbf{r}_2)$$

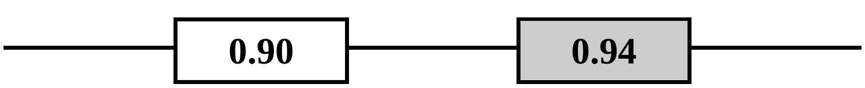
$$\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_1 \times \mathbf{r}_2$$

In general, $1 - (1-\mathbf{r}_1) \times (1-\mathbf{r}_2) \times \ldots \times (1-\mathbf{r}_k)$

7. Compute the reliability of the following system of independent components (the numbers represent the reliability of each component):

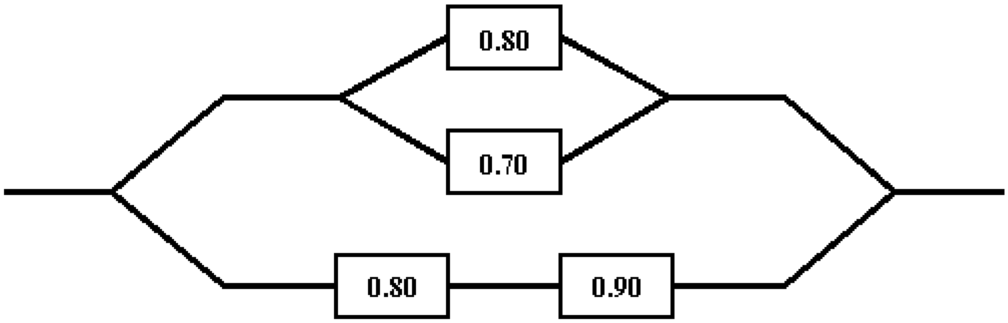


$1 - (1 - 0.80) \times (1 - 0.70) = 0.94$ OR $0.80 + 0.70 - 0.80 \times 0.70 = 0.94$.



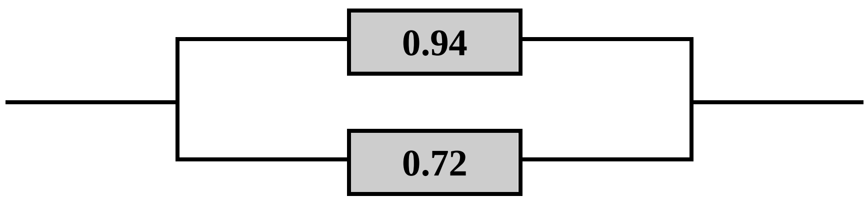
$0.90 \times 0.94 =$ **0.846**.

8. Compute the reliability of the following system of independent components (the numbers represent the reliability of each component):



$1 - (1 - 0.80) \times (1 - 0.70) = 0.94$ OR $0.80 + 0.70 - 0.80 \times 0.70 = 0.94$.

$0.80 \times 0.90 = 0.72$.



$1 - (1 - 0.94) \times (1 - 0.72) = \mathbf{0.9832}$ OR $0.80 + 0.70 - 0.80 \times 0.70 = \mathbf{0.9832}$.