STAT 400
UIUC

Example 2.3 (Bonus)

Stepanov
Dalpiaz

1. Let X be a discrete random variable with p.m.f. $p(x)$ that is positive on the odd non-negative integers $\{1, 3, 5, 7, 9, \dots\}$ and is zero elsewhere. Suppose

$$p(1) = c \quad (\text{unknown}), \qquad p(k) = \frac{1}{2^k}, \quad k = 3, 5, 7, 9, \dots .$$

a) Find the value of c that makes this is a valid probability distribution.

Must have $\sum_{\text{all } x} p(x) = 1.$

$$1 = p(1) + p(3) + p(5) + p(7) + p(9) + \dots$$

$$= c + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \frac{1}{2^9} + \dots = c + \frac{\frac{1}{8}}{1 - \frac{1}{4}} = c + \frac{1}{6}.$$

$$\Rightarrow \qquad c = \mathbf{\frac{5}{6}}.$$

b) Find $\mathrm{E}(X)$.

$$\mathrm{E}(X) = \sum_{\text{all } x} x \cdot p(x) = \frac{5}{6} + \frac{3}{2^3} + \frac{5}{2^5} + \frac{7}{2^7} + \frac{9}{2^9} + \dots$$

$$\frac{1}{4}\mathrm{E}(X) = \qquad \frac{5}{24} + \frac{3}{2^5} + \frac{5}{2^7} + \frac{7}{2^9} + \dots$$

$$\Rightarrow \qquad \frac{3}{4}\mathrm{E}(X) = \frac{5}{6} + \frac{3}{8} - \frac{5}{24} + \frac{2}{2^5} + \frac{2}{2^7} + \frac{2}{2^9} + \dots = 1 + \frac{\frac{2}{32}}{1 - \frac{1}{4}} = \frac{13}{12}.$$

$$\Rightarrow \qquad \mathrm{E}(X) = \mathbf{\frac{13}{9}}.$$

OR

$$M_X(t) = E(e^{tX}) = e^{1t} \cdot \frac{5}{6} + \sum_{k=1}^{\infty} e^{(2k+1)t} \cdot \frac{1}{2^{(2k+1)}}$$

$$= \frac{5}{6} \cdot e^t + \frac{e^t}{2} \cdot \sum_{k=1}^{\infty} \left(\frac{e^{2t}}{4} \right)^k = \frac{5}{6} \cdot e^t + \frac{e^t}{2} \cdot \frac{\frac{e^{2t}}{4}}{1 - \frac{e^{2t}}{4}}$$

$$= \frac{5}{6} \cdot e^t + \frac{e^t}{2} \cdot \frac{e^{2t}}{4 - e^{2t}} = \frac{5}{6} \cdot e^t + \frac{e^{3t}}{8 - 2e^{2t}}, \qquad t < \ln 2.$$

$$M'_X(t) = \frac{5}{6} \cdot e^t + \frac{3e^{3t}\left(8 - 2e^{2t}\right) - e^{3t}\left(-4e^{2t}\right)}{\left(8 - 2e^{2t}\right)^2} = \frac{5}{6} \cdot e^t + \frac{24e^{3t} - 2e^{5t}}{\left(8 - 2e^{2t}\right)^2},$$

$$t < \ln 2.$$

$$E(X) = M'_X(0) = \frac{5}{6} + \frac{22}{36} = \frac{52}{36} = \mathbf{\frac{13}{9}}.$$

OR

$$M_X(t) = E(e^{tX}) = e^{1t} \cdot \frac{5}{6} + \sum_{k=1}^{\infty} e^{(2k+1)t} \cdot \frac{1}{2^{(2k+1)}}$$

$$= \frac{5}{6} \cdot e^t + \frac{e^t}{2} \cdot \sum_{k=1}^{\infty} \left(\frac{e^{2t}}{4} \right)^k = \frac{5}{6} \cdot e^t + \frac{e^t}{2} \cdot \left(\frac{1}{1 - \frac{e^{2t}}{4}} - 1 \right)$$

$$= \frac{1}{3} \cdot e^t + \frac{4e^t}{8 - 2e^{2t}}, \qquad t < \ln 2.$$

$$M'_X(t) = \frac{1}{3} \cdot e^t + \frac{4e^t\left(8 - 2e^{2t}\right) - 4e^t\left(-4e^{2t}\right)}{\left(8 - 2e^{2t}\right)^2} = \frac{1}{3} \cdot e^t + \frac{32e^t + 8e^{3t}}{\left(8 - 2e^{2t}\right)^2},$$

$$t < \ln 2.$$

$$E(X) = M'_X(0) = \frac{1}{3} + \frac{40}{36} = \frac{52}{36} = \mathbf{\frac{13}{9}}.$$