STAT 400
UIUC

Examples for 2.4, 2.5

Stepanov
Dalpiaz

Binomial Distribution:

1. The number of trials, n, is fixed.
2. Each trial has two possible outcomes: "success" and "failure".
3. The probability of "success", p, is the same from trial to trial.
4. The trials are independent.
5. X = number of "successes" in n independent trials.

Then

$$P(X = k) = \binom{n}{k} \cdot p^{k} \cdot (1 - p)^{n-k} = {}_{n}C_{k} \cdot p^{k} \cdot (1 - p)^{n-k},$$

where $k = 0, 1, \ldots, n$.

$E(X) = n \cdot p$ $\qquad$ $Var(X) = n \cdot p \cdot (1 - p)$ $\qquad$ $SD(X) = \sqrt{n \cdot p \cdot (1 - p)}$

1. Bart Simpson takes a multiple choice exam in his Statistics 101 class. The exam has 15 questions, each has 5 possible answers, only one of which is correct. Bart did not study for the exam, so he guesses independently on every question. Let X denote the number of questions that Bart gets right.

a) Is it appropriate to use Binomial model for this problem?

Yes. Binomial, $n = 15$, $p = {}^{1}\!/_{5} = 0.20$.

b) What is the expected number of questions that Bart would get right?

$E(X) = n \cdot p = 15 \cdot 0.20 =$ **3**.

c) What is the probability that Bart answers exactly 3 questions correctly?

$P(X = 3) = \binom{15}{3} \cdot (0.20)^{3} \cdot (0.80)^{12} =$ **0.2501389**.

OR

P(X = 3) = CDF @ 3 – CDF @ 2 = 0.648 – 0.398 = **0.250**.

d) What is the probability that Bart would get at most 5 of the questions right?

$P(X \leq 5) = \text{CDF @ } 5 = \mathbf{0.939}$.

e) What is the probability that Bart would get more than half of the questions right (i.e. what is the probability that Bart would get at least 8 of the questions right)?

$P(X \geq 8) = 1 - \text{CDF @ } 7 = 1 - 0.996 = \mathbf{0.004}$.

f) Find the probability that Bart answers between 4 and 6 (including both 4 and 6) questions correctly?

$P(4 \leq X \leq 6) = \text{CDF @ } 6 - \text{CDF @ } 3 = 0.982 - 0.648 = \mathbf{0.334}$.

OR

$$P(4 \leq X \leq 6) = \binom{15}{4} \cdot (0.20)^4 \cdot (0.80)^{11} + \binom{15}{5} \cdot (0.20)^5 \cdot (0.80)^{10} + \binom{15}{6} \cdot (0.20)^6 \cdot (0.80)^9$$

$$= 0.18760417 + 0.10318229 + 0.04299262 = \mathbf{0.33377908}.$$

Binomial

$n = 15$

$p = 0.20$

Outcome	Probability
0	0.03518437
1	0.13194140
2	0.23089744
3	0.25013890
4	0.18760417
5	0.10318229
6	0.04299262
7	0.01381906
8	0.00345476
9	0.00067176
10	0.00010076
11	0.00001145
12	0.00000095
13	0.00000006
14	0.00000000
15	0.00000000

k	CDF at k
0	0.03518437
1	0.16712577
2	0.39802321
3	0.64816210
4	0.83576628
5	0.93894857
6	0.98194119
7	0.99576025
8	0.99921501
9	0.99988677
10	0.99998754
11	0.99999899
12	0.99999994
13	1.00000000
14	1.00000000

EXCEL: =BINOMDIST($k, n, p, 0$) =BINOMDIST($k, n, p, 1$)

2. ☺ An automobile salesman thinks that the probability of making a sale is 0.30. If he talks to five customers on a particular day, what is the probability that he will make exactly 2 sales? (Assume independence.)

X = number of sales. Binomial, $n = 5$, $p = 0.30$.

P(X = 2) = $\binom{5}{2} \cdot (0.30)^2 \cdot (0.70)^3$ = **0.3087**.

OR

P(X = 2) = CDF @ 2 – CDF @ 1 = 0.837 – 0.528 = **0.309**.

2½. *Often On time Parcel Service* (*OOPS*) delivers a package to the wrong address with probability 0.05 on any delivery. Suppose that each delivery is independent of all the others. There were 7 packages delivered on a particular day.

a) ☺ What is the probability that at least one of them was delivered to the wrong address?

P(X ≥ 1) = 1 – CDF @ 0 = 1 – 0.698 = **0.302**.

b) What is the probability that exactly two of them were delivered to the wrong address?

P(X = 2) = $\binom{7}{2} \cdot (0.05)^2 \cdot (0.95)^5$ = **0.0406235**.

OR

P(X = 2) = CDF @ 2 – CDF @ 1 = 0.996 – 0.956 = **0.040**.

c) What is the probability that at most two of them were delivered to the wrong address?

P(X ≤ 2) = CDF @ 2 = **0.996**.

d) What is the probability that at least two of them were delivered to the wrong address?

P(X ≥ 2) = 1 – CDF @ 1 = 1 – 0.956 = **0.044**.

3. ☺ A major oil company has decided to drill independent test wells in the Alaskan wilderness. The probability of any well producing oil is 0.30. Find the probability that the fifth well is the first to produce oil.

$$\begin{array}{ccccccccc} F & & F & & F & & F & & S \\ 0.70 & \cdot & 0.70 & \cdot & 0.70 & \cdot & 0.70 & \cdot & 0.30 \end{array} = \mathbf{0.07203}.$$

Geometric Distribution:

X = the number of **independent** trials until the first "success".

Then

$$P(X = x) = (1-p)^{x-1} \cdot p, \qquad x = 1, 2, 3, \ldots .$$

$$E(X) = \frac{1}{p}. \qquad \mathrm{Var}(X) = \frac{1-p}{p^2}.$$

4. A slot machine at a casino randomly rewards 15% of the attempts. Assume that all attempts are independent.

a) What is the probability that your first reward occurs on your fourth trial?

$$\begin{array}{ccccccc} F & & F & & F & & S \\ 0.85 & \cdot & 0.85 & \cdot & 0.85 & \cdot & 0.15 \end{array} \approx \mathbf{0.092}.$$

Geometric, $p = 0.15$.

b) What is the probability that your first reward occurs on your seventh trial?

$$\begin{array}{ccccccccccccc} F & & F & & F & & F & & F & & F & & S \\ 0.85 & \cdot & 0.85 & \cdot & 0.85 & \cdot & 0.85 & \cdot & 0.85 & \cdot & 0.85 & \cdot & 0.15 \end{array} \approx \mathbf{0.0566}.$$

Geometric, $p = 0.15$.

c) What is the probability that your get three rewards in ten trials?

Binomial, $n = 10$, $p = 0.15$.

$$\mathrm{P}(X = 3) = \binom{10}{3} \cdot (0.15)^3 \cdot (0.85)^7 \approx \mathbf{0.1298}.$$

d) What is the probability that your third reward occurs on your tenth trial?

[9 trials: 2 S's & 7 F's] S

$$\left[\binom{9}{2} \cdot (0.15)^2 \cdot (0.85)^7 \right] \cdot 0.15 \approx \mathbf{0.03895}.$$

Negative Binomial, $k = 3$, $p = 0.15$.

Negative Binomial Distribution:

X = the number of **independent** trials until the k^{th} "success".

Then

$$\mathrm{P}(X = x) = \binom{x-1}{k-1} \cdot p^k \cdot (1-p)^{x-k}, \qquad x = k, k+1, k+2, \ldots .$$

$$\mathrm{E}(X) = \frac{k}{p}. \qquad \mathrm{V}(X) = \frac{k \cdot (1-p)}{p^2}.$$

EXCEL: $=\text{NEGBINOMDIST}(x-k, k, p)$ gives $\mathrm{P}(X = x)$

e) What is the probability that your fourth reward occurs on your fifteenth trial?

[14 trials: 3 S's & 11 F's] S

$$\left[\binom{14}{3} \cdot (0.15)^3 \cdot (0.85)^{11} \right] \cdot 0.15 \approx \mathbf{0.030837}.$$

Negative Binomial, $k = 4$, $p = 0.15$.

f) What is the probability that your get four rewards in fifteen trials?

Binomial, $n = 15$, $p = 0.15$.

$$\mathrm{P}(X = 4) = \binom{15}{4} \cdot (0.15)^4 \cdot (0.85)^{11} \approx \mathbf{0.11564}.$$

= =

Let X be a random variable with a Geometric distribution with probability of "success" p.
Then

$$\mathrm{P}(\mathrm{X} > a) = \sum_{k=a+1}^{\infty} (1-p)^{k-1} p = \frac{(1-p)^a p}{1-(1-p)} = (1-p)^a, \qquad a = 0, 1, 2, 3, \ldots .$$

OR

X = number of independent attempts needed to get the first "success".

$$\mathrm{P}(\mathrm{X} > a) = \mathrm{P}(\text{the first } a \text{ attempts are "failures"}) = (1-p)^a, \qquad a = 0, 1, 2, 3, \ldots .$$

Ex. Let X denote the number of rolls of a fair 6-seded die needed to observe the first "6".

$$\mathrm{P}(4 \le \mathrm{X} \le 7) = \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^6\left(\frac{1}{6}\right) \approx 0.299622.$$

OR

$$\mathrm{P}(4 \le \mathrm{X} \le 7) = \mathrm{P}(\mathrm{X} > 3) - \mathrm{P}(\mathrm{X} > 7) = \left(\frac{5}{6}\right)^3 - \left(\frac{5}{6}\right)^7 \approx 0.299622.$$

For positive integers a and b, $\qquad \mathrm{P}(\mathrm{X} > a + b \mid \mathrm{X} > a) = \mathrm{P}(\mathrm{X} > b)$

(***memoryless*** property).

$$\mathrm{P}(\mathrm{X} > a + b \mid \mathrm{X} > a) = \frac{\mathrm{P}(\mathrm{X} > a+b \cap \mathrm{X} > a)}{\mathrm{P}(\mathrm{X} > a)} = \frac{\mathrm{P}(\mathrm{X} > a+b)}{\mathrm{P}(\mathrm{X} > a)}$$

$$= \frac{(1-p)^{a+b}}{(1-p)^a} = (1-p)^b = \mathrm{P}(\mathrm{X} > b).$$