Examples for 2.6

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Poisson Distribution:

X = the number of occurrences of a particular event in an interval of time or space.

$$P(X = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, 3, \ldots .$$

$$E(X) = \lambda, \qquad \mathrm{Var}(X) = \lambda.$$

Table III (pp. 580 – 582) gives $P(X \le x)$

1. Traffic accidents at a particular intersection follow Poisson distribution with an average rate of 1.4 per week.

a) What is the probability that the next week is accident-free?

1 week $\Rightarrow$ $\lambda = 1.4$. $P(X = 0) = \dfrac{1.4^0 \cdot e^{-1.4}}{0!} \approx \mathbf{0.2466}$.

b) What is the probability that there will be exactly 3 accidents next week?

1 week $\Rightarrow$ $\lambda = 1.4$. $P(X = 3) = \dfrac{1.4^3 \cdot e^{-1.4}}{3!} \approx \mathbf{0.1128}$.

c) What is the probability that there will be at most 2 accidents next week?

1 week $\Rightarrow$ $\lambda = 1.4$.

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{1.4^0 \cdot e^{-1.4}}{0!} + \frac{1.4^1 \cdot e^{-1.4}}{1!} + \frac{1.4^2 \cdot e^{-1.4}}{2!}$$

$$\approx 0.2466 + 0.3452 + 0.2417 = \mathbf{0.8335}.$$

d) What is the probability that there will be at least 2 accidents during the next two weeks?

2 weeks $\Rightarrow$ $\lambda = 2.8$.

$$P(X \geq 2) = 1 - [\,P(X=0) + P(X=1)\,] = 1 - \left[\frac{2.8^{0} \cdot e^{-2.8}}{0!} + \frac{2.8^{1} \cdot e^{-2.8}}{1!}\right]$$

$$\approx 1 - [\,0.0608 + 0.1703\,] = \mathbf{0.7689}.$$

e) What is the probability that there will be exactly 5 accidents during the next four weeks?

4 weeks $\Rightarrow$ $\lambda = 5.6$.

$$P(X = 5) = \frac{5.6^{5} \cdot e^{-5.6}}{5!} \approx \mathbf{0.1697}.$$

f) What is the probability that there will be exactly 2 accidents tomorrow?

1 day $\Rightarrow$ $\lambda = 0.2$.

$$P(X = 2) = \frac{0.2^{2} \cdot e^{-0.2}}{2!} \approx \mathbf{0.0164}.$$

g) What is the probability that the next accident will not occur for three days?

3 days $\Rightarrow$ $\lambda = 0.6$.

$$P(X = 0) = \frac{0.6^{0} \cdot e^{-0.6}}{0!} \approx \mathbf{0.5488}.$$

h) What is the probability that there will be exactly three accident-free weeks during the next eight weeks?

"Success" = an accident-free week

1 week $\Rightarrow$ $\lambda = 1.4$.

$$p = P(\text{"Success"}) = P(X = 0) = \frac{1.4^{0} \cdot e^{-1.4}}{0!} \approx 0.2466.$$

$$P(\text{exactly 3 accident-free weeks in 8 weeks}) = {}_{8}C_{3} \cdot 0.2466^{3} \cdot 0.7534^{5} \approx \mathbf{0.20384}.$$

(Binomial distribution)

i) What is the probability that there will be exactly five accident-free days during the next week?

"Success" = an accident-free day

1 day $\Rightarrow$ $\lambda = 0.2.$ $p = \text{P}(\text{"Success"}) = \text{P}(\text{X} = 0) = \dfrac{0.2^{\,0} \cdot e^{-0.2}}{0\,!} \approx 0.81873.$

$\text{P}(\text{exactly 5 accident-free days in 7 days}) = {}_7\text{C}_5 \cdot 0.81873^5 \cdot 0.18127^2 \approx \mathbf{0.25385}.$

(Binomial distribution)

When n is large ($n \ge 20$) and p is small ($p \le 0.05$) and $n \cdot p \le 5$, Binomial probabilities can be approximated by Poisson probabilities.
For this, set $\lambda = n \cdot p$.

2. Suppose the defective rate at a particular factory is 1%. Suppose 50 parts were selected from the daily output of parts. Let X denote the number of defective parts in the sample.

a) Find the probability that the sample contains exactly 2 defective parts.

$$\text{P}(X = 2) = \binom{50}{2} \cdot (0.01)^2 \cdot (0.99)^{48} \approx \mathbf{0.075618}.$$

b) Use Poisson approximation to find the probability that the sample contains exactly 2 defective parts.

$$\lambda = n \cdot p = 0.5.$$

$$\text{P}(X = 2) = \frac{0.5^{\,2} \cdot e^{-0.5}}{2\,!} \approx \mathbf{0.075816}.$$

c) Find the probability that the sample contains at most 1 defective part.

$$\mathrm{P}(X \le 1) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1)$$

$$= \binom{50}{0} \cdot (0.01)^0 \cdot (0.99)^{50} + \binom{50}{1} \cdot (0.01)^1 \cdot (0.99)^{49} \approx \mathbf{0.910565}.$$

d) Use Poisson approximation to find the probability that the sample contains at most 1defective part.

$$\mathrm{P}(X \le 1) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1)$$

$$= \frac{0.5^0 \cdot e^{-0.5}}{0!} + \frac{0.5^1 \cdot e^{-0.5}}{1!} \approx \mathbf{0.909796}.$$