STAT 400
UIUC

Discrete vs Continuous RVs

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random variables

discrete	continuous
probability **mass** function	probability **density** function
p.m.f.	p.d.f.
$p(x) = \mathrm{P}(\mathrm{X} = x)$	$f(x)$
$\forall x \quad 0 \le p(x) \le 1$	$\forall x \quad f(x) \ge 0$
$\sum_{\text{all } x} p(x) = 1$	$\int_{-\infty}^{\infty} f(x)\,dx = 1$

cumulative distribution function

c.d.f.

$$\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$$

$$\mathrm{F}(x) = \sum_{y \le x} p(y) \qquad\qquad \mathrm{F}(x) = \int_{-\infty}^{x} f(y)\,dy$$

expected value

$$\mathrm{E}(\mathrm{X}) = \mu_{\mathrm{X}}$$

discrete	continuous
$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot p(x)$	$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

discrete	continuous
$\mathrm{E}(g(\mathrm{X})) = \sum_{\text{all } x} g(x) \cdot p(x)$	$\mathrm{E}(g(\mathrm{X})) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\,dx$

variance

$$\mathrm{Var}(X) = \sigma_X^2 = E([X-\mu_X]^2) = E(X^2) - [E(X)]^2$$

discrete

$$\mathrm{Var}(X) = \sum_{\text{all } x} (x-\mu_X)^2 \cdot p(x)$$

$$= \sum_{\text{all } x} x^2 \cdot p(x) - [E(X)]^2$$

continuous

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f(x)\,dx$$

$$= \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx\right] - [E(X)]^2$$

moment-generating function

$$M_X(t) = E(e^{tX})$$

discrete

$$M_X(t) = \sum_{\text{all } x} e^{tx} \cdot p(x)$$

continuous

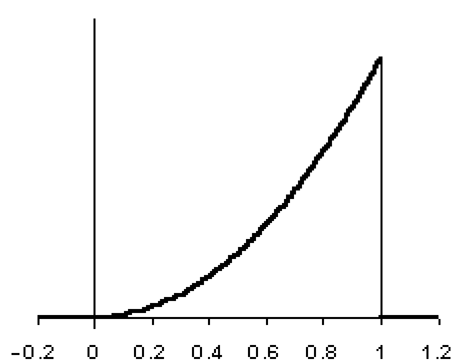
$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

Example 1:

Let X be a continuous random variable with the probability density function

$f(x) = k \cdot x^2$, $0 < x < 1$,

$f(x) = 0$, otherwise.



a) What must the value of k be so that $f(x)$ is a probability density function?

1) $f(x) \geq 0$,

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 k \cdot x^2\,dx = k \cdot \int_0^1 x^2\,dx$$

$$= k \cdot \left(\frac{x^3}{3} \right) \Bigg|_0^1 = k \cdot \left(\frac{1}{3} \right) = \frac{k}{3}. \quad \Rightarrow \quad k = \mathbf{3}.$$

b) Find the cumulative distribution function $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$.

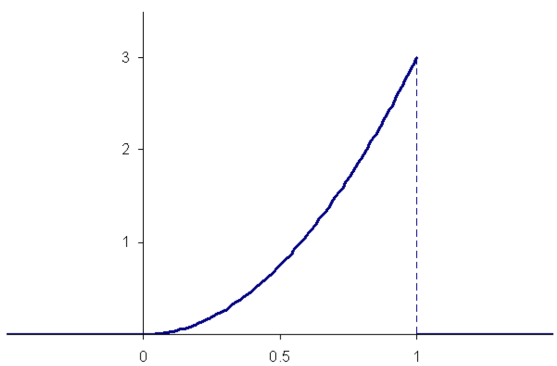
$$f_{\mathrm{X}}(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$x < 0$ $\quad \mathrm{F}_{\mathrm{X}}(x) = 0.$

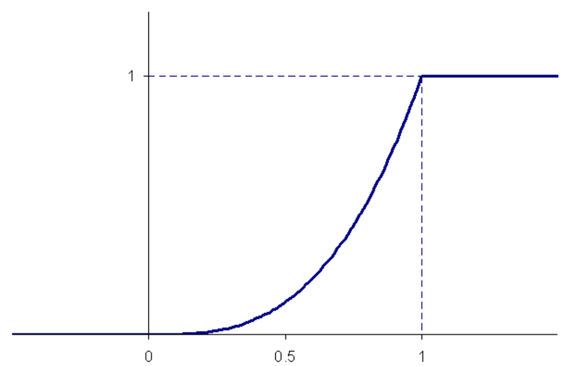
$0 \le x < 1$ $\quad \mathrm{F}_{\mathrm{X}}(x) = \int_0^x 3y^2\,dy = x^3.$

$x \ge 1$ $\quad \mathrm{F}_{\mathrm{X}}(x) = 1.$

$f_{\mathrm{X}}(x)$



$\mathrm{F}_{\mathrm{X}}(x)$

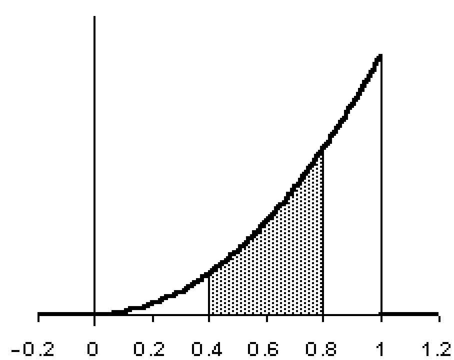


c) Find the probability $\mathrm{P}(0.4 \le \mathrm{X} \le 0.8)$.

$$\mathrm{P}(0.4 \le \mathrm{X} \le 0.8) = \int_{0.4}^{0.8} f(x)\,dx$$

$$= \int_{0.4}^{0.8} 3 \cdot x^2\,dx = x^3 \Big|_{0.4}^{0.8}$$

$$= 0.8^3 - 0.4^3 = \mathbf{0.448}.$$



OR

$P(0.4 \le X \le 0.8) = F_X(0.8) - F_X(0.4\text{-}) = 0.8^3 - 0.4^3 = \mathbf{0.448}.$

d) Find the median of the distribution of X.

Need $m = ?$ such that $(\text{Area to the left of } m) = \int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}.$

$$\frac{1}{2} = \int_{-\infty}^{m} f(x)\,dx = \int_{0}^{m} 3 \cdot x^2\,dx = x^3 \Big|_{0}^{m} = m^3.$$

$m = \sqrt[3]{1/2} = \mathbf{0.7937}.$

e) Find $\mu_X = E(X)$.

$$E(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{0}^{1} x \cdot \left(3 \cdot x^2\right)dx = 3 \cdot \int_{0}^{1} x^3\,dx.$$

$$= 3 \cdot \left(\frac{x^4}{4}\right)\Bigg|_{0}^{1} = \mathbf{\frac{3}{4}} = \mathbf{0.75}.$$

f) Find $\sigma_X = SD(X)$.

$$Var(X) = \sigma_X^2 = \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx\right] - \left(\mu_X\right)^2 = \left[\int_{0}^{1} 3 \cdot x^4\,dx\right] - \left(\frac{3}{4}\right)^2$$

$$= 3 \cdot \left(\frac{x^5}{5}\right)\Bigg|_{0}^{1} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80} = 0.0375.$$

$\sigma_X = SD(X) = \sqrt{Var(X)} = \sqrt{0.0375} = \mathbf{0.19365}.$

g) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx = \int_0^1 e^{tx} \cdot 3x^2\,dx.$$

$u = 3x^2,$ $dv = e^{tx}\,dx,$

$du = 6x\,dx,$ $v = \frac{1}{t} e^{tx}.$

$$M_X(t) = \int_0^1 e^{tx} \cdot 3x^2\,dx = \left(3x^2 \cdot \frac{1}{t} e^{tx}\right)\Big|_0^1 - \int_0^1 \left(\frac{1}{t} e^{tx} \cdot 6x\right) dx$$

$$= \frac{3}{t} e^t - \int_0^1 \left(\frac{1}{t} e^{tx} \cdot 6x\right) dx$$

$u = 6x,$ $dv = = \frac{1}{t} e^{tx}\,dx,$

$du = 6\,dx,$ $v = \frac{1}{t^2} e^{tx}.$

$$M_X(t) = \frac{3}{t} e^t - \int_0^1 \left(\frac{1}{t} e^{tx} \cdot 6x\right) dx = \frac{3}{t} e^t - \left(6x \cdot \frac{1}{t^2} e^{tx}\right)\Big|_0^1 - \int_0^1 \left(\frac{1}{t^2} e^{tx} \cdot 6\right) dx$$

$$= \frac{3}{t} e^t - \frac{6}{t^2} e^t + \left(\frac{6}{t^3} e^{tx}\right)\Big|_0^1 = \frac{3}{t} e^t - \frac{6}{t^2} e^t + \frac{6}{t^3} e^t - \frac{6}{t^3}, \qquad t \neq 0.$$

$M_X(0) = 1.$

h) Find $E(\sqrt{X})$ and $E(\ln X)$.

$$E(\sqrt{X}) = \int_0^1 \sqrt{x} \cdot 3x^2\,dx = \frac{6}{7}. \qquad E(\ln X) = \int_0^1 \ln x \cdot 3x^2\,dx = -\frac{1}{3}.$$

Example 2:

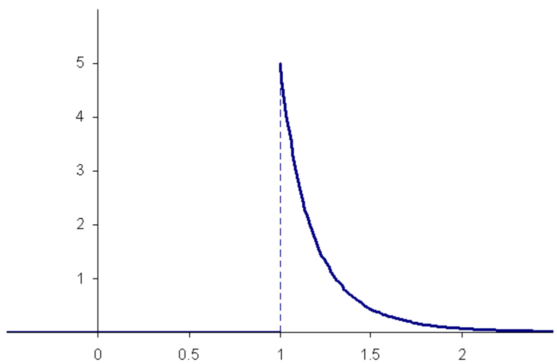
$$f_X(x) = \begin{cases} 5x^{-6} & x > 1 \\ 0 & \text{o.w.} \end{cases}$$

$x < 1 \qquad F_X(x) = 0.$

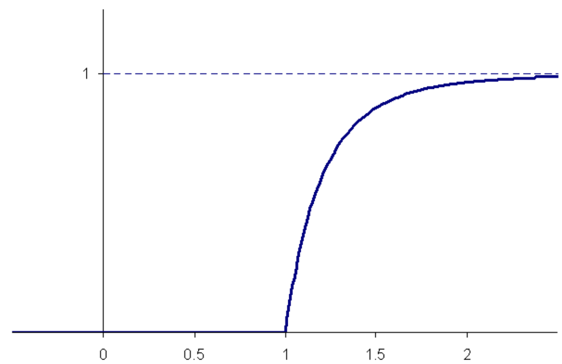
$x \ge 1 \qquad F_X(x) = \int_1^x 5y^{-6}\,dy$

$$= -y^{-5}\Big|_1^x = 1 - x^{-5}.$$

$f_X(x)$



$F_X(x)$



$$E(X) = \mu_X = \int_1^\infty x \cdot 5x^{-6}\,dx = \int_1^\infty 5x^{-5}\,dx = \frac{5}{4} = 1.25.$$

$$E(X^2) = \int_1^\infty x^2 \cdot 5x^{-6}\,dx = \int_1^\infty 5x^{-4}\,dx = \frac{5}{3}.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{5}{3} - \left(\frac{5}{4}\right)^2 = \frac{5}{48}.$$

$E(X^{10})$ does NOT exist since $\int_1^\infty x^{10} \cdot 5x^{-6}\,dx$ diverges.

Median: $\quad F_X(m) = \frac{1}{2}. \qquad 1 - m^{-5} = \frac{1}{2}. \qquad m = \sqrt[5]{2} \approx 1.1487.$

30th percentile: $\quad F_X(\pi_{0.30}) = 0.30. \qquad 1 - (\pi_{0.30})^{-5} = 0.30.$

$$\pi_{0.30} = \sqrt[5]{\frac{1}{0.70}} \approx 1.07394.$$

Example 3:

Suppose a random variable X has the following probability density function:

$$f(x) = \begin{cases} C \cdot e^{-x} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) What must the value of C be so that $f(x)$ is a probability density function?

For $f(x)$ to be a probability density function, we must have:

1) $f(x) \ge 0$, 2) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$.

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^1 C \cdot e^{-x}\,\mathrm{d}x = C \cdot \int_0^1 e^{-x}\,\mathrm{d}x$$

$$= C \cdot \left(-e^{-x}\right)\Big|_0^1 = C \cdot \left(1 - e^{-1}\right) = C \cdot \left(\frac{e-1}{e}\right).$$

Therefore, $C = \left(\frac{e}{e-1}\right) \approx \mathbf{1.5819767}$.

$$f(x) = \begin{cases} \left(\frac{e}{e-1}\right) \cdot e^{-x} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

b) Find $\mu_X = E(X)$.

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,\mathrm{d}x = \int_0^1 x \cdot \left(\frac{e}{e-1}\right) \cdot e^{-x}\,\mathrm{d}x = \left(\frac{e}{e-1}\right) \cdot \int_0^1 x \cdot e^{-x}\,\mathrm{d}x.$$

Integrating by parts,

$$\int_0^1 x \cdot e^{-x}\,\mathrm{d}x = \left[(x) \cdot \left(-e^{-x}\right)\right]\Big|_0^1 - \int_0^1 \left(-e^{-x}\right)\mathrm{d}x$$

$$= -e^{-1} + \int_0^1 e^{-x}\,\mathrm{d}x = -e^{-1} + \left(-e^{-x}\right)\Big|_0^1 = 1 - 2 \cdot e^{-1} = \frac{e-2}{e}.$$

Therefore,

$$\mu_X = E(X) = \left(\frac{e}{e-1}\right)\cdot\int_0^1 x\cdot e^{-x}\,dx = \left(\frac{e}{e-1}\right)\cdot\left(\frac{e-2}{e}\right) = \frac{e-2}{e-1} \approx \mathbf{0.418}.$$

c) Find the cumulative distribution function $F(x) = P(X \le x)$.

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy.$$

$F(x) = 0$ for $x < 0$.

$F(x) = 1$ for $x > 1$.

For $0 \le x \le 1$,

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_0^x \left(\frac{e}{e-1}\right)\cdot e^{-y}\,dy = \left(\frac{e}{e-1}\right)\cdot\left(-e^{-y}\right)\Big|_0^x$$

$$= \left(\frac{e}{e-1}\right)\cdot\left(1-e^{-x}\right)$$

$$= \left(\frac{e}{e-1}\right)\cdot\left(\frac{e^x-1}{e^x}\right).$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \left(\frac{e}{e-1}\right)\cdot\left(\frac{e^x-1}{e^x}\right) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

d) Find the median of the probability distribution of X.

Need $m = ?$ such that $P(X \le m) = P(X \ge m) = 1/2$.

Thus, $\frac{1}{2} = F(m) = \left(\frac{e}{e-1}\right)\cdot\left(\frac{e^m-1}{e^m}\right).$

$\Rightarrow \quad e^m - 1 = \left(\frac{e-1}{2\cdot e}\right)\cdot e^m. \qquad \Rightarrow \quad e^m - \left(\frac{e-1}{2\cdot e}\right)\cdot e^m = 1.$

$\Rightarrow \quad \left(\frac{e+1}{2\cdot e}\right)\cdot e^m = 1. \qquad \Rightarrow \quad e^m = \frac{2\cdot e}{e+1}.$

$\Rightarrow \quad m = \ln\left(\frac{2\cdot e}{e+1}\right) \approx \mathbf{0.3799}.$

e) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = E(e^{tX}) = \int_0^1 e^{tx} \cdot \left(\frac{e}{e-1}\right) \cdot e^{-x}\, dx = \left(\frac{e}{e-1}\right) \cdot \int_0^1 e^{(t-1)x}\, dx$$

$$= \left(\frac{e}{e-1}\right) \cdot \left(\frac{1}{t-1} \cdot e^{(t-1)x}\right)\Bigg|_0^1 = \left(\frac{e}{e-1}\right) \cdot \frac{1}{t-1} \cdot \left(e^{t-1} - 1\right)$$

$$= \frac{e^t - e}{(e-1)\cdot(t-1)}, \qquad t \neq 1.$$

$$M_X(1) = \frac{e}{e-1}.$$

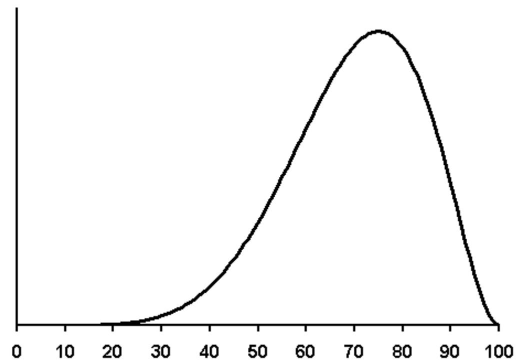
f) Find $E(2^X)$.

$$E(2^X) = E(e^{\ln 2 \cdot X}) = M_X(\ln 2) = \frac{e^{\ln 2} - e}{(e-1)\cdot(\ln 2 - 1)} = \frac{e-2}{(e-1)\cdot(1-\ln 2)}.$$

Example 4:

A simple model for describing mortality in the general population in a particular country is given by the probability density function

$$f(y) = \frac{252}{10^{18}}\ y^6\,(100-y)^2, \qquad 0 < y < 100.$$



a) Verify that $f(y)$ is a valid probability density function.

1. $f(y) \ge 0$ for each y; ✓

2. $\int\limits_{-\infty}^{\infty} f(y)\,dy = 1.$

$$\int\limits_{-\infty}^{\infty} f(y)\,dy = \int\limits_{0}^{100} \frac{252}{10^{18}}\ y^6\,(100-y)^2\,dy = \int\limits_{0}^{1} 252\ x^6\,(1-x)^2\,dx$$

$$= 252\cdot\left[\frac{1}{7}x^7 - 2\cdot\frac{1}{8}x^8 + \frac{1}{9}x^9\right]\Bigg|_0^1 = 252\cdot\frac{2}{504} = 1. \qquad ✓$$

b) Based on this model, which event is more likely

A: a person dies between the ages of 70 and 80

or

B: a person lives past age 80?

A: $$\int\limits_{70}^{80} \frac{252}{10^{18}}\ y^6\,(100-y)^2\,dy = \int\limits_{0.7}^{0.8} 252\ x^6\,(1-x)^2\,dx$$

$$= 252\cdot\left[\frac{1}{7}x^7 - 2\cdot\frac{1}{8}x^8 + \frac{1}{9}x^9\right]\Bigg|_{0.7}^{0.8}$$

$$\approx 0.7382 - 0.4628 = 0.2754.$$

B: $$\int_{80}^{100} \frac{252}{10^{18}} \; y^6 (100-y)^2 \, dy = \int_{0.8}^{1.0} 252 \; x^6 (1-x)^2 \, dx$$

$$= 252 \cdot \left[\frac{1}{7} x^7 - 2 \cdot \frac{1}{8} x^8 + \frac{1}{9} x^9 \right] \Bigg|_{0.8}^{1.0}$$

$$\approx 1 - 0.7382 = 0.2618.$$

A is more likely.

c) Given that a randomly selected individual just celebrated his 60th birthday, find the probability that he will live past age 80.

$$\mathrm{P}(\text{over } 80 \mid \text{over } 60) = \frac{\mathrm{P}(\text{over } 80 \cap \text{over } 60)}{\mathrm{P}(\text{over } 60)} = \frac{\displaystyle\int_{80}^{100} \frac{252}{10^{18}} \; y^6 (100-y)^2 \, dy}{\displaystyle\int_{60}^{100} \frac{252}{10^{18}} \; y^6 (100-y)^2 \, dy}$$

$$\approx \frac{1-0.7382}{1-0.2318} = \frac{0.2618}{0.7682} \approx \mathbf{0.3408}.$$

d) Find the value of y that maximizes $f(y)$ (**mode**).

$$f'(y) = \frac{252}{10^{18}} \left[6\,y^5 (100-y)^2 - 2\,y^6 (100-y) \right]$$

$$= \frac{252}{10^{18}} \; y^5 (100-y) [\, 6(100-y) - 2\,y \,]$$

$$= \frac{252}{10^{18}} \; y^5 (100-y) [\, 600 - 8\,y \,] = 0.$$

$\Rightarrow$ $y = 0$, $y = 100$ (not max), $y = \mathbf{75}$ years (max).

e) Find the (average) life expectancy.

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f(y)\,dy = \int_{0}^{100} \frac{252}{10^{18}}\ y^7 (100-y)^2\,dy = \int_{0}^{1} 252 \cdot 100\ x^7 (1-x)^2\,dx$$

$$= 252 \cdot 100 \cdot \left[\frac{1}{8} x^8 - 2 \cdot \frac{1}{9} x^9 + \frac{1}{10} x^{10} \right] \Bigg|_{0}^{1} = 252 \cdot 100 \cdot \frac{2}{720} = \mathbf{70} \text{ years.}$$

OR

Consider $X = \frac{Y}{100}$. Then $Y = 100\,X$, and X has the probability density function

$$f(x) = 252\ x^6 (1-x)^2, \quad 0 < x < 1.$$

Then X has Beta distribution with $\alpha = 7$ and $\beta = 3$.

$E(X) = \frac{\alpha}{\alpha+\beta} = \frac{7}{7+3} = 0.70.$ $E(Y) = 100\,E(X) = \mathbf{70}$ years.

f) Find the standard deviation of the lifetimes.

$\text{Var}(X) = \frac{7 \cdot 3}{11 \cdot 10^2} = \frac{21}{1100}.$ $\text{Var}(Y) = 100^2\,\text{Var}(X) = \frac{2100}{11}.$

$\text{SD}(Y) \approx 13.817.$

OR

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 \cdot f(y)\,dy = \int_{0}^{100} \frac{252}{10^{18}}\ y^8 (100-y)^2\,dy = \ldots$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \ldots$$

$$\text{SD}(Y) = \sqrt{\text{Var}(Y)} = \ldots$$

Example 5:

Let Y denote a random variable with probability density function given by

$$f(y) = \frac{1}{2} e^{-|y|}, \qquad -\infty < y < \infty. \qquad (\text{double exponential p.d.f.})$$

a) Find the moment-generating function of Y. For which values of t does it exist?

$$M_Y(t) = \int_{-\infty}^{\infty} e^{ty} \cdot \frac{1}{2} e^{-|y|} dy = \int_{-\infty}^{0} e^{ty} \cdot \frac{1}{2} e^{-|y|} dy + \int_{0}^{\infty} e^{ty} \cdot \frac{1}{2} e^{-|y|} dy$$

$$= \int_{-\infty}^{0} e^{ty} \cdot \frac{1}{2} e^{y} dy + \int_{0}^{\infty} e^{ty} \cdot \frac{1}{2} e^{-y} dy$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{y(t+1)} dy + \frac{1}{2} \int_{0}^{\infty} e^{y(t-1)} dy$$

Note that the first integral converges only if $t + 1 > 0$,

and the second integral converges only if $t - 1 < 0$.

Therefore, the moment-generating function is only defined for $-1 < t < 1$.

$$M_Y(t) = \frac{1}{2(t+1)} e^{y(t+1)} \Big|_{-\infty}^{0} + \frac{1}{2(t-1)} e^{y(t-1)} \Big|_{0}^{\infty} = \frac{1}{2(t+1)} - \frac{1}{2(t-1)}$$

$$= \frac{(t-1)-(t+1)}{2(t+1)(t-1)} = \frac{-2}{2\left(t^2-1\right)} = \frac{1}{1-t^2}, \qquad -1 < t < 1.$$

b) Find E(Y).

$$M_Y'(t) = -(1-t^2)^{-2}(-2t) = 2t(1-t^2)^{-2}$$

$$\Rightarrow \qquad E(Y) = M_Y'(0) = \mathbf{0}.$$

OR

$$E(Y) = \int_{-\infty}^{\infty} y \cdot \frac{1}{2} e^{-|y|} dy = \mathbf{0}, \qquad \text{since } y \cdot \frac{1}{2} e^{-|y|} \text{ is an odd function.}$$

c) Find $\operatorname{Var}(Y)$.

$$M_Y''(t) = 2(1-t^2)^{-2} + 2t(-2)(1-t^2)^{-3}(-2t) = \frac{2+6t^2}{\left(1-t^2\right)^3}$$

$$\Rightarrow \quad E(Y^2) = M_Y''(0) = 2. \qquad \Rightarrow \quad \operatorname{Var}(Y) = 2 - 0^2 = \mathbf{2}.$$

OR

$$\operatorname{Var}(Y) = E(Y^2) - [E(Y)]^2 = \int_{-\infty}^{\infty} y^2 \cdot \frac{1}{2} e^{-|y|} dy = \ldots = \mathbf{2}.$$

d) Find the cumulative distribution function $F(y) = P(Y \le y)$.

If $y < 0$, $$F(y) = \int_{-\infty}^{y} \frac{1}{2} e^{-|x|} dx = \int_{-\infty}^{y} \frac{1}{2} e^{x} dx = \frac{1}{2} e^{y}.$$

If $y \ge 0$, $$F(y) = \int_{-\infty}^{y} \frac{1}{2} e^{-|x|} dx = \int_{-\infty}^{0} \frac{1}{2} e^{x} dx + \int_{0}^{y} \frac{1}{2} e^{-x} dx$$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - e^{-y}\right) = 1 - \frac{1}{2} e^{-y}.$$

Therefore,

$$F(y) = \begin{cases} \frac{1}{2} e^{y} & y < 0 \\ 1 - \frac{1}{2} e^{-y} & y \ge 0 \end{cases}$$

e) Find $\mathrm{E}(\mathrm{Y}^k)$ for positive integer k.

$$\mathrm{E}(\mathrm{Y}^k) = \int_{-\infty}^{\infty} y^k \cdot \frac{1}{2} e^{-|y|}\, dy = \int_{-\infty}^{0} y^k \cdot \frac{1}{2} e^{y}\, dy + \int_{0}^{\infty} y^k \cdot \frac{1}{2} e^{-y}\, dy = \ldots$$

k odd $\qquad \ldots = 0.$

k even $\qquad \ldots = 2 \cdot \int_{0}^{\infty} y^k \cdot \frac{1}{2} e^{-y}\, dy = \int_{0}^{\infty} y^k \cdot e^{-y}\, dy = \Gamma(k+1) = k!.$

OR

Taylor Formula:

$$\mathrm{M}_{\mathrm{Y}}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathrm{M}_{\mathrm{Y}}^{(k)}(0) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathrm{E}(\mathrm{Y}^k).$$

On the other hand,

$$\mathrm{M}_{\mathrm{Y}}(t) = \frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n}.$$

$\Rightarrow$ If k odd, $\qquad \mathrm{E}(\mathrm{Y}^k) = 0.$

If k even, $\qquad k = 2n, \qquad \mathrm{E}(\mathrm{Y}^k) = k!.$