STAT 400
UIUC

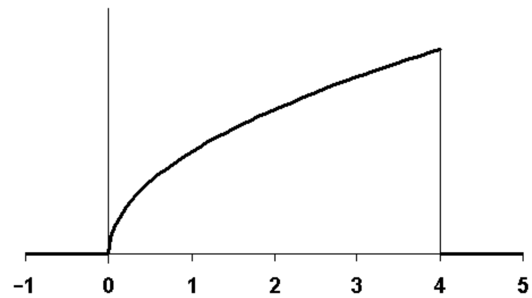
Answers for 3.1

Stepanov
Dalpiaz

1. Let X be a continuous random variable with the probability density function

$$f(x) = k \cdot \sqrt{x}, \quad 0 \le x \le 4,$$

$$f(x) = 0, \quad \text{otherwise.}$$



a) What must the value of k be so that $f(x)$ is a probability density function?

1) $f(x) \ge 0,$ 2) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1.$

$$1 = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_0^4 k \cdot \sqrt{x}\,\mathrm{d}x = k \cdot \int_0^4 x^{1/2}\,\mathrm{d}x$$

$$= k \cdot \left(\frac{2}{3} \cdot x^{3/2} \right) \Bigg|_0^4 = k \cdot \left(\frac{16}{3} \right). \quad \Rightarrow \quad k = \mathbf{\frac{3}{16}} = \mathbf{0.1875}.$$

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy.$$

$x \le 0$ $\quad F_X(x) = 0.$

$0 \le x \le 4$ $\quad F_X(x) = \int_0^x \frac{3}{16} \cdot \sqrt{y}\,dy = \frac{3}{16} \cdot \left(\frac{2}{3} \cdot y^{3/2} \right) \Bigg|_0^x = \frac{1}{8} \cdot x^{3/2}.$

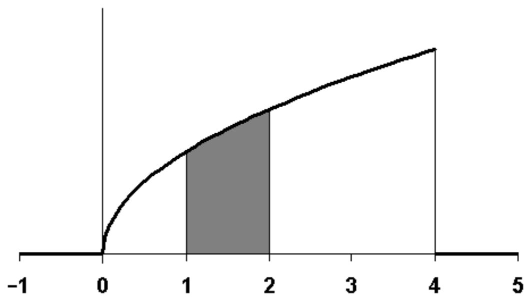
$x \ge 4$ $\quad F_X(x) = 1.$

c) Find the probability $P(1 \le X \le 2)$.

$$P(1 \le X \le 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{3}{16} \cdot \sqrt{x}\,dx = \frac{3}{16} \cdot \left(\frac{2}{3} \cdot x^{3/2}\right)\Big|_1^2$$

$$\approx \mathbf{0.22855}.$$



OR

$$P(1 \le X \le 2) = F_X(2) - F_X(1-) = \frac{1}{8} \cdot 2^{3/2} - \frac{1}{8} \cdot 1^{3/2} \approx \mathbf{0.22855}.$$

d) Find the median of the distribution of X. That is, find m such that $P(X \le m) = P(X \ge m) = 1/2$.

$$\frac{1}{2} = \int_{-\infty}^{m} f(x)\,dx = \int_0^m \frac{3}{16} \cdot \sqrt{x}\,dx = \frac{3}{16} \cdot \left(\frac{2}{3} \cdot x^{3/2}\right)\Big|_0^m = \frac{1}{8} \cdot m^{3/2}.$$

$$4 = m^{3/2}. \qquad m = \sqrt[3]{4^2} \approx \mathbf{2.51984}.$$

e) Find the 30th percentile of the distribution of X. That is, find a such that $P(X \le a) = 0.30$.

$$0.30 = \int_{-\infty}^{a} f(x)\,dx = \int_0^a \frac{3}{16} \cdot \sqrt{x}\,dx = \frac{3}{16} \cdot \left(\frac{2}{3} \cdot x^{3/2}\right)\Big|_0^a = \frac{1}{8} \cdot a^{3/2}.$$

$$2.4 = a^{3/2}. \qquad a = \sqrt[3]{2.4^2} \approx \mathbf{1.79256}.$$

f) Find $\mu_X = E(X)$.

$$E(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^4 x \cdot \left(\frac{3}{16} \cdot \sqrt{x}\right) dx = \frac{3}{16} \cdot \int_0^4 x^{3/2}\,dx.$$

$$= \frac{3}{16} \cdot \left(\frac{2}{5} \cdot x^{5/2}\right)\Bigg|_0^4 = \mathbf{2.4}.$$

g) Find $\sigma_X = SD(X)$.

$$Var(X) = \sigma_X^2 = \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx\right] - (\mu_X)^2 = \left[\int_0^4 \frac{3}{16} \cdot x^{5/2}\,dx\right] - (2.4)^2$$

$$= \frac{3}{16} \cdot \left(\frac{2}{7} \cdot x^{7/2}\right)\Bigg|_0^4 - (2.4)^2 = 1.09714.$$

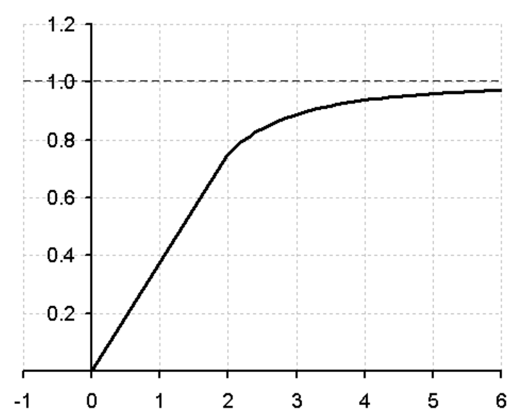
$$\sigma_X = SD(X) = \sqrt{Var(X)} = \sqrt{1.09714} = \mathbf{1.04745}.$$

2. Let X be a continuous random variable with the cumulative distribution function

$F(x) = 0, \qquad x < 0,$

$F(x) = \frac{3}{8} \cdot x, \qquad 0 \le x \le 2,$

$F(x) = 1 - \frac{1}{x^2}, \qquad x > 2.$



a) Find the probability density function $f(x)$.

$f(x) = F'(x)$.

$x < 0 \qquad f(x) = F'(x) = 0,$

$0 \le x \le 2 \qquad f(x) = F'(x) = \dfrac{3}{8},$

$x > 2 \qquad f(x) = F'(x) = \dfrac{2}{x^3}.$

b) Find the probability $P(1 \le X \le 4)$.

$$P(1 \le X \le 4) = F(4) - F(1-) = \frac{15}{16} - \frac{3}{8} = \mathbf{\frac{9}{16}} = \mathbf{0.5625}.$$

OR

$$P(1 \le X \le 4) = \int_1^4 f(x)\,dx = \int_1^2 \frac{3}{8}\,dx + \int_2^4 \frac{2}{x^3}\,dx = \mathbf{0.5625}.$$

c) Find $\mu_X = E(X)$.

$$E(X) = \mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^2 x \cdot \left(\frac{3}{8}\right) dx + \int_2^{\infty} x \cdot \left(\frac{2}{x^3}\right) dx.$$

$$= \frac{3}{8} \cdot \left(\frac{x^2}{2}\right) \Bigg|_0^2 + 2 \cdot \left(-\frac{1}{x}\right) \Bigg|_2^{\infty} = \frac{3}{4} + 1 = \mathbf{\frac{7}{4}} = \mathbf{1.75}.$$

d) Find $\sigma_X = SD(X)$.

$$Var(X) = \sigma_X^2 = \left[\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx\right] - (\mu_X)^2.$$

However, $\displaystyle\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^2 x^2 \cdot \frac{3}{8}\,dx + \int_2^{\infty} x^2 \cdot \frac{2}{x^3}\,dx$ diverges.

Therefore, $Var(X)$ and $SD(X)$ are not finite.