STAT 400
UIUC

Examples for 3.3

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1. At *Initech*, the salaries of the employees are normally distributed with mean μ = \$36,000 and standard deviation σ = \$5,000.

a) Mr. Smith is paid \$42,000. What proportion of the employees of *Initech* are paid less that Mr. Smith?

$$P(X < 42{,}000) = P\left(Z < \frac{42{,}000 - 36{,}000}{5{,}000}\right) = P(Z < 1.20) = \mathbf{0.8849}.$$

b) What proportion of the employees have their salaries over \$40,000?

$$P(X > 40{,}000) = P\left(Z > \frac{40{,}000 - 36{,}000}{5{,}000}\right) = P(Z > 0.80) = 1 - 0.7881 = \mathbf{0.2119}.$$

c) Suppose 10 *Initech* employees are randomly and independently selected. What is the probability that 3 of them have their salaries over \$40,000?

Let Y = number of employees (out of 10) who have salaries over \$40,000.
Then Y has Binomial distribution, $n = 10$, $p = \mathbf{0.2119}$ (see (b)).

$$P(Y = 3) = {}_{10}C_3 \cdot (0.2119)^3 \cdot (0.7881)^7 = \mathbf{0.2156}.$$

d) What proportion of the employees have their salaries between \$30,000 and \$40,000?

$$P(30{,}000 < X < 40{,}000) = P\left(\frac{30{,}000 - 36{,}000}{5{,}000} < Z < \frac{40{,}000 - 36{,}000}{5{,}000}\right)$$

$$= P(-1.2 < Z < 0.80) = 0.7881 - 0.1151 = \mathbf{0.6730}.$$

e) Mrs. Jones claims that her salary is high enough to just put her among the highest paid 15% of all employees working at *Initech*. Find her salary.

Need $x = ?$ such that $P(X > x) = 0.15.$ (area to the right is 0.15)

First, need $z = ?$ such that $P(Z > z) = 0.15.$

$z = 1.04.$

$X = \mu + \sigma Z.$ $x = 36{,}000 + 5{,}000 \times 1.04 =$ **$41,200**.

f) Ms. Green claims that her salary is so low that 90% of the employees make more than she does. Find her salary.

Need $x = ?$ such that $P(X > x) = 0.90.$ (area to the right is 0.90)

First, need $z = ?$ such that $P(Z > z) = 0.90.$

$z = -1.28.$

$X = \mu + \sigma Z.$ $x = 36{,}000 + 5{,}000 \times (-1.28) =$ **$29,600**.

2. Suppose that the lifetime of ***Outlast*** batteries is normally distributed with mean $\mu = 240$ hours and unknown standard deviation. Suppose also that 20% of the batteries last less than 219 hours. Find the standard deviation of the distribution of the lifetimes.

Need $\sigma = ?$ Know $P(X < 219) = 0.20.$

First, need $z = ?$ such that $P(Z < z) = 0.20.$

$z = -0.84.$

$X = \mu + \sigma Z.$ $219 = 240 + \sigma \times (-0.84).$

$-21 = \sigma \times (-0.84).$

$\sigma =$ **25** hours.

Let X be normally distributed with mean μ and standard deviation σ.
Then the moment-generating function of X is

$$M_X(t) = e^{\mu t + \sigma^2 t^2 / 2}.$$

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \int_{-\infty}^{\infty} e^{t(\mu + \sigma z)} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = e^{\mu t + \sigma^2 t^2/2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z - \sigma t)^2/2} dz$$

$$= e^{\mu t + \sigma^2 t^2/2},$$

since $\frac{1}{\sqrt{2\pi}} e^{-(z-\sigma t)^2/2}$ is the probability density function of a $N(\sigma t, 1)$ random variable.

Let $Y = a X + b$. Then $M_Y(t) = e^{bt} M_X(at)$.

Therefore, Y is normally distributed with mean $a\mu + b$ and variance $a^2 \sigma^2$ (standard deviation $|a|\sigma$).

1. (continued)

g) All *Initech* employees receive a memo instructing them to put away 4% of their salaries plus \$100 per month (\$1,200 per year) in a special savings account to supplement Social Security. What proportion of the employees would put away more than \$3,000 per year?

$Y = 0.04\,X + 1{,}200.$ $\qquad P(Y > 3{,}000) = ?$

$Y > 3{,}000 \quad \Leftrightarrow \quad X > 45{,}000.$

$$P(X > 45{,}000) = P\left(Z > \frac{45{,}000 - 36{,}000}{5{,}000}\right) = P(Z > 1.80) = 1 - 0.9641 = \mathbf{0.0359}.$$

OR

$\mu_Y = 0.04 \times 36{,}000 + 1{,}200 = \$2{,}640$, $\sigma_Y = 0.04 \times 5{,}000 = \$200$.

$$P(Y > 3{,}000) = P\left(Z > \frac{3{,}000 - 2{,}640}{200}\right) = P(Z > 1.80) = 1 - 0.9641 = \mathbf{0.0359}.$$

3. Suppose the average daily temperature [in degrees Fahrenheit] in June in Anytown is a random variable T with mean $\mu_T = 85$ and standard deviation $\sigma_T = 7$. The daily air conditioning cost Q, in dollars, for Anytown State University, is related to T by

$$Q = 120\,T + 750.$$

Suppose that T is a normal random variable. Compute the probability that the daily air conditioning cost on a typical June day for the university will exceed $12,210.

Q has Normal distribution.

$\mu_Q = 120\,\mu_T + 750 = 120 \cdot 85 + 750 = \$10{,}950$.

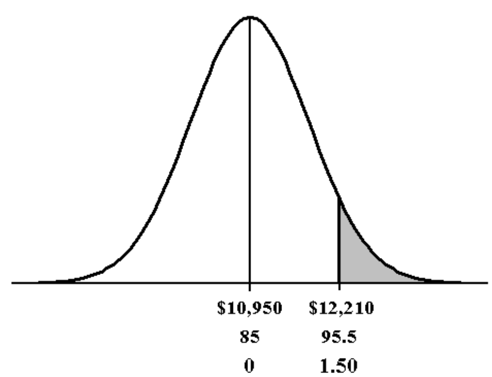
$\sigma_Q^2 = (120)^2 \cdot \sigma_T^2 = (120)^2 \cdot 7^2 = 840^2$. $\sigma_Q = \$840$.

$$P(Q > 12{,}210) = P\left(Z > \frac{12{,}210 - 10{,}950}{840}\right) = P(Z > 1.50) = 1 - \Phi(1.50) = 1 - 0.9332 = \mathbf{0.0668}.$$

OR

$12{,}210 = 120\,T + 750$. $\Leftrightarrow$ $T = 95.5$.

$$\begin{aligned} P(Q > 12{,}210) &= P(T > 95.5) \\ &= P\left(Z > \frac{95.5 - 85}{7}\right) \\ &= P(Z > 1.50) \\ &= 1 - \Phi(1.50) \\ &= 1 - 0.9332 \\ &= \mathbf{0.0668}. \end{aligned}$$



4. Consider a random variable X with the moment-generating function

$$M_X(t) = e^{3t + 8t^2} = \exp(3t + 8t^2).$$

a) Find $P(X > 0)$.

Normal distribution: $M_X(t) = e^{\mu t + \sigma^2 t^2 / 2}$.

$\Rightarrow$ X has a Normal distribution with $\mu = 3$ and $\sigma^2/2 = 8$.

$\Rightarrow$ $E(X) = \mu = 3,$ $\quad$ $Var(X) = \sigma^2 = 16.$ $\quad$ $\sigma = 4.$

$$P(X > 0) = P\left(Z > \frac{0-3}{4}\right) = P(Z > -0.75) = 1 - \Phi(-0.75) = 1 - 0.2266 = \mathbf{0.7734}.$$

a) Find $P(-1 < X < 9)$.

$$P(-1 < X < 9) = P\left(\frac{-1-3}{4} < Z < \frac{9-3}{4}\right) = P(-1.00 < Z < 1.50)$$

$$= \Phi(1.50) - \Phi(-1.00) = 0.9332 - 0.1587 = \mathbf{0.7745}.$$

EXCEL:

= NORMSDIST(z)	gives	$P(Z \le z) = \Phi(z)$
= NORMSINV(p)	gives	z such that $P(Z \le z) = p$
= NORMDIST($x, \mu, \sigma, 1$)	gives	$P(X \le x)$, where X is $N(\mu, \sigma^2)$
= NORMDIST($x, \mu, \sigma, 0$)	gives	$f(x)$, p.d.f. of $N(\mu, \sigma^2)$
= NORMINV(p, μ, σ)	gives	x such that $P(X \le x) = p$, where X is $N(\mu, \sigma^2)$

5.* Show that the odd moments of $N(0, \sigma^2)$ are zero and the even moments are

$$\mu_{2n} = \frac{(2n)!\sigma^{2n}}{2^n (n)!}$$

Taylor Formula:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} M^{(r)}(0) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r .$$

Since X is $N(0, \sigma^2)$,

$$M_X(t) = \exp\left\{\frac{\sigma^2 t^2}{2}\right\} = \sum_{n=0}^{\infty} \frac{\sigma^{2n} t^{2n}}{2^n n!}$$

Therefore,

if r is odd, $\mu_r = 0$,

if $r = 2n$ is even, $\frac{\sigma^{2n}}{2^n n!} = \frac{1}{r!} \mu_r \quad \Rightarrow \quad \mu_{2n} = \frac{(2n)!\sigma^{2n}}{2^n (n)!}.$

OR

Def $\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} \, du, \quad x > 0$

$$\Gamma(x) = (x-1)\,\Gamma(x-1)$$

$\Gamma(n) = (n-1)!$ if n is an integer

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\mu_{2n} = \int_{-\infty}^{\infty} x^{2n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}\, dx = \int_{0}^{\infty} x^{2n} \frac{2}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}\, dx = \ldots$$

$$u = \frac{x^2}{2\sigma^2} \qquad du = \frac{x\,dx}{\sigma^2} \qquad dx = \frac{du\,\sigma}{\sqrt{2u}}$$

$$\ldots = \int_{0}^{\infty} 2^n \sigma^{2n} \frac{1}{\sqrt{\pi}} u^{n-1/2} e^{-u}\, du = 2^n \sigma^{2n} \frac{1}{\sqrt{\pi}} \Gamma\left(n+\frac{1}{2}\right).$$

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\cdot\left(n-\frac{3}{2}\right)\cdot\left(n-\frac{5}{2}\right)\cdot\ldots\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$= \frac{1}{2^n}\cdot(2n-1)\cdot(2n-3)\cdot(2n-5)\cdot\ldots\cdot 3\cdot 1\cdot\sqrt{\pi}$$

$$= \frac{1}{2^n}\cdot\frac{(2n)!}{(2n)\cdot(2n-2)\cdot(2n-4)\cdot\ldots\cdot 4\cdot 2}\cdot\sqrt{\pi}$$

$$= \frac{1}{2^n}\cdot\frac{(2n)!}{2^n\cdot(n)!}\cdot\sqrt{\pi}$$

$$\Rightarrow \qquad \mu_{2n} = \frac{(2n)!\,\sigma^{2n}}{2^n\,(n)!}.$$