STAT 400
UIUC

Examples for 4.1, 4.4 (1)

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Multivariate Distributions

Let X and Y be two discrete random variables. The **joint probability mass function** $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x, y) = \mathrm{P}(\mathrm{X} = x \text{ and } \mathrm{Y} = y).$$

Let A be any set consisting of pairs of (x, y) values. Then

$$\mathrm{P}((\mathrm{X}, \mathrm{Y}) \in A) = \sum_{(x,y)\in A}\sum p(x, y).$$

Let X and Y be two continuous random variables. Then $f(x, y)$ is the **joint probability density function** for X and Y if for any two-dimensional set A

$$\mathrm{P}((\mathrm{X}, \mathrm{Y}) \in A) = \iint_A f(x, y)\, dx\, dy.$$

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

x \ y	0	1	2	
1	0.15	0.10	0	
2	0.25	0.30	0.20	

a) Find $\mathrm{P}(\mathrm{X} > \mathrm{Y})$.

$\mathrm{P}(\mathrm{X} > \mathrm{Y}) = p(1, 0) + p(2, 0) + p(2, 1) = 0.15 + 0.25 + 0.30 = \mathbf{0.70}$.

b) Find $\mathrm{P}(\mathrm{X} + \mathrm{Y} = 2)$.

$\mathrm{P}(\mathrm{X} + \mathrm{Y} = 2) = p(1, 1) + p(2, 0) = 0.10 + 0.25 = \mathbf{0.35}$.

The **marginal probability mass functions** of X and of Y are given by

$$p_X(x) = \sum_{\text{all } y} p(x, y), \qquad p_Y(y) = \sum_{\text{all } x} p(x, y).$$

The **marginal probability density functions** of X and of Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx.$$

c) Find the (marginal) probability distributions $p_X(x)$ of X and $p_Y(y)$ of Y.

x	$p_X(x)$
1	0.25
2	0.75

y	$p_Y(y)$
0	0.40
1	0.40
2	0.2

If $p(x, y)$ is the joint probability mass function of (X, Y) OR $f(x, y)$ is the joint probability density function of (X, Y), then

discrete

$$E(g(X, Y)) = \sum_{\text{all } x} \sum_{\text{all } y} g(x, y) \cdot p(x, y)$$

continuous

$$E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y)\,dx\,dy$$

d) Find $E(X)$, $E(Y)$, $E(X + Y)$, $E(X \cdot Y)$.

$E(X) = 1 \times 0.25 + 2 \times 0.75 =$ **1.75**.

$E(Y) = 0 \times 0.40 + 1 \times 0.40 + 2 \times 0.20 =$ **0.8**.

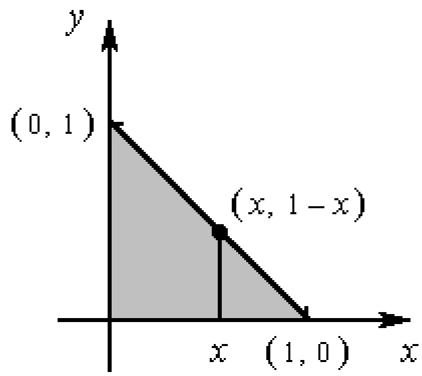
$E(X + Y) = 1 \times 0.15 + 2 \times 0.25 + 2 \times 0.10 + 3 \times 0.30 + 3 \times 0 + 4 \times 0.20 =$ **2.55**.

OR

$E(X + Y) = E(X) + E(Y) = 1.75 + 0.8 =$ **2.55**.

$E(X \cdot Y) = 0 \times 0.15 + 0 \times 0.25 + 1 \times 0.10 + 2 \times 0.30 + 2 \times 0 + 4 \times 0.20 =$ **1.5**.

2. Alexis Nuts, Inc. markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. Suppose the net weight of each can is exactly 1 lb, but the weight contribution of each type of nut is random. Because the three weights sum to 1, a joint probability model for any two gives all necessary information about the weight of the third type. Let X = the weight of almonds in a selected can and Y = the weight of cashews.



Then the region of positive density is D = $\{(x, y) : 0 \le x \le 1, 0 \le y \le 1, x + y \le 1\}$.

Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Verify that $f(x,y)$ is a legitimate probability density function.

1. $f(x,y) \ge 0$ for all (x,y). ✓

2. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^{1-x} 60\,x^2\,y\,dy \right) dx = \int_0^1 \left(30\,x^2\,(1-x)^2 \right) dx$$

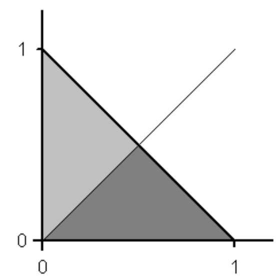
$$= \int_0^1 \left(30\,x^2 - 60\,x^3 + 30\,x^4 \right) dx = \left(10\,x^3 - 15\,x^4 + 6\,x^5 \right) \Big|_0^1 = 1.$$ ✓

b) Find the probability that the two types of nuts together make up less than 50% of the can. That is, find the probability P(X + Y < 0.50). (Find the probability that peanuts make up over 50% of the can.)

$$P(X + Y < 0.50) = \int_0^{0.5} \left(\int_0^{0.5-x} 60\,x^2\,y\,dy \right) dx = \int_0^{0.5} 30\,x^2\,(0.5 - x)^2\,dx$$

$$= \int_0^{0.5} \left(7.5\,x^2 - 30\,x^3 + 30\,x^4 \right) dx = \left(2.5\,x^3 - 7.5\,x^4 + 6\,x^5 \right) \Big|_0^{0.5} = \mathbf{\frac{1}{32}} = \mathbf{0.03125}.$$

c) Find the probability that there are more almonds than cashews in a can. That is, find the probability $\mathrm{P}(\mathrm{X} > \mathrm{Y})$.



$$\mathrm{P}(\mathrm{X} > \mathrm{Y}) = \int_0^{1/2} \left(\int_y^{1-y} 60\,x^2\, y\, dx \right) dy$$

$$= \int_0^{1/2} 20\,y \left(\int_y^{1-y} 3\,x^2\, dx \right) dy$$

$$= \int_0^{1/2} 20\,y \left((1-y)^3 - y^3 \right) dy$$

$$= \int_0^{1/2} 20\,y \left(1 - 3\,y + 3\,y^2 - 2\,y^3 \right) dx = \int_0^{1/2} \left(20\,y - 60\,y^2 + 60\,y^3 - 40\,y^4 \right) dx$$

$$= \left(10\,y^2 - 20\,y^3 + 15\,y^4 - 8\,y^5 \right) \Big|_0^{1/2} = \mathbf{\frac{11}{16}} = \mathbf{0.6875}.$$

OR

$$\mathrm{P}(\mathrm{X} > \mathrm{Y}) = 1 - \int_0^{1/2} \left(\int_x^{1-x} 60\,x^2\, y\, dy \right) dx = 1 - \int_0^{1/2} 30\,x^2 \left(\int_x^{1-x} 2\,y\, dy \right) dx$$

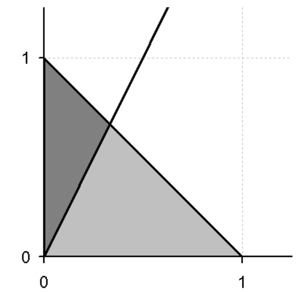
$$= 1 - \int_0^{1/2} 30\,x^2 \left((1-x)^2 - x^2 \right) dx = 1 - \int_0^{1/2} 30\,x^2 \left(1 - 2\,x \right) dx$$

$$= 1 - \int_0^{1/2} \left(30\,x^2 - 60\,x^3 \right) dx = 1 - \left(10\,x^3 - 15\,x^4 \right) \Big|_0^{1/2} = \mathbf{\frac{11}{16}} = \mathbf{0.6875}.$$

OR

$$\mathrm{P}(\mathrm{X} > \mathrm{Y}) = \int_0^{1/2} \left(\int_0^{x} 60\,x^2\, y\, dy \right) dx + \int_{1/2}^{1} \left(\int_0^{1-x} 60\,x^2\, y\, dy \right) dx = \ldots$$

d) Find the probability that there are at least twice as many cashews as there are almonds. That is, find the probability $P(2X \le Y)$.



$$P(Y \ge 2X) = \int_0^{1/3} \left(\int_{2x}^{1-x} 60\,x^2\,y\,dy \right) dx$$

$$= \int_0^{1/3} \left(30\,x^2 \left[(1-x)^2 - (2x)^2 \right] \right) dx$$

$$= \int_0^{1/3} \left(30\,x^2 - 60\,x^3 - 90\,x^4 \right) dx$$

$$= \int_0^{1/3} \left(30\,x^2 - 60\,x^3 - 90\,x^4 \right) dx = \left(10\,x^3 - 15\,x^4 - 18\,x^5 \right) \Big|_0^{1/3}$$

$$= \frac{10}{27} - \frac{15}{81} - \frac{18}{243} = \mathbf{\frac{1}{9}}.$$

e) Find the marginal probability density function for X.

$$f_X(x) = \int_0^{1-x} 60\,x^2\,y\,dy = 30\,x^2 \int_0^{1-x} 2\,y\,dy = 30\,x^2\,(1-x)^2, \qquad 0 < x < 1.$$

f) Find the marginal probability density function for Y.

$$f_Y(y) = \int_0^{1-y} 60\,x^2\,y\,dx = 20\,y \int_0^{1-y} 3\,x^2\,dx = 20\,y\,(1-y)^3, \qquad 0 < y < 1.$$

g) Find $E(X)$, $E(Y)$, $E(X+Y)$, $E(X \cdot Y)$.

$$E(X) = \int_0^1 x \cdot 30\,x^2\,(1-x)^2\,dx = \int_0^1 \left(30\,x^3 - 60\,x^4 + 30\,x^5 \right) dx$$

$$= \left(7.5\,x^4 - 12\,x^5 + 5\,x^6 \right) \Big|_0^1 = \mathbf{0.5} = \mathbf{\frac{1}{2}}.$$

$$E(Y) = \int_0^1 y \cdot 20\,y(1-y)^3\,dy = \int_0^1 \left(20\,y^2 - 60\,y^3 + 60\,y^4 - 20\,y^5\right)dy$$

$$= \left(\frac{20}{3} y^3 - 15\,y^4 + 12\,y^5 - \frac{20}{6} y^6 \right) \Bigg|_0^1 = \mathbf{\frac{1}{3}}.$$

$$E(X+Y) = E(X) + E(Y) = \mathbf{\frac{5}{6}}.$$

$$E(X \cdot Y) = \int_0^1 \left(\int_0^{1-x} x\,y \cdot 60\,x^2\,y\,dy \right) dx = \int_0^1 \left(20\,x^3\,(1-x)^3\right) dx$$

$$= \int_0^1 \left(20\,x^3 - 60\,x^4 + 60\,x^5 - 20\,x^6\right) dx$$

$$= \left(5\,x^4 - 12\,x^5 + 10\,x^6 - \frac{20}{7} x^7 \right) \Bigg|_0^1 = \mathbf{\frac{1}{7}}.$$

h) If 1 lb of almonds costs the company \$1.00, 1 lb of cashews costs \$1.50, and 1 lb of peanuts costs \$0.60, what is the expected total cost of the content of a can?

$$\text{Total cost} = (1.00)\,X + (1.50)\,Y + (0.60)\,(1 - X - Y) = 0.6 + 0.4\,X + 0.9\,Y.$$

$$E(\text{Total cost}) = 0.6 + 0.4\,E(X) + 0.9\,E(Y) = 0.60 + 0.40 \cdot \frac{1}{2} + 0.90 \cdot \frac{1}{3} = \mathbf{\$1.10}.$$

OR

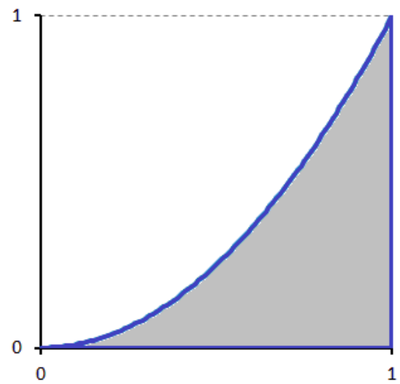
$$E(\text{Total cost}) = (1.00)\,E(X) + (1.50)\,E(Y) + (0.60)\,E(1 - X - Y)$$

$$= 1.00 \cdot \frac{1}{2} + 1.50 \cdot \frac{1}{3} + 0.60 \cdot \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \mathbf{\$1.10}.$$

2¼. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} C\,x\,y^3 & 0 < x < 1,\ 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.



$$1 = \int_0^1 \left(\int_0^{x^2} C\,x\,y^3\,dy \right) dx$$

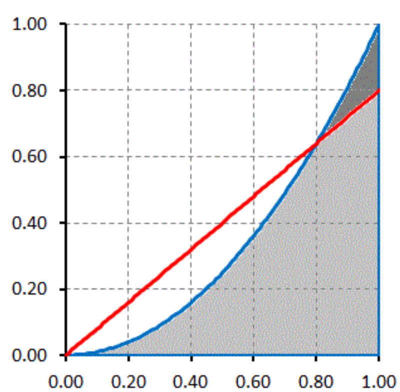
$$= \int_0^1 \frac{C}{4}\,x\,y^4 \Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 \frac{C}{4}\,x^9\,dx = \frac{C}{40}\,x^{10}\Big|_0^1 = \frac{C}{40}.$$

$$\Rightarrow \quad C = \mathbf{40}.$$

b) Find $P(5Y > 4X)$.

$$P(5Y > 4X) = P(Y > 0.80\,X)$$



$y = 0.80\,x$ and $y = x^2$

$\Rightarrow \quad x = 0.80$

$$= \int_{0.80}^{1} \left(\int_{0.80x}^{x^2} 40\,x\,y^3\,dy \right) dx$$

$$= \int_{0.80}^{1} 10\,x\left(y^4\right)\Big|_{0.80x}^{x^2} dx$$

$$= \int_{0.80}^{1} \left(10\,x^9 - 4.096\,x^5\right) dx$$

$$= \left(x^{10} - \frac{256}{375}\,x^6 \right)\Big|_{0.80}^{1} \approx 0.3889.$$

OR

$$\int_{0.64}^{0.80}\left(\int_{\sqrt{y}}^{1.25y} 40\,x\,y^3\,dx\right)dy + \int_{0.80}^{1}\left(\int_{\sqrt{y}}^{1} 40\,x\,y^3\,dx\right)dy = \ldots$$

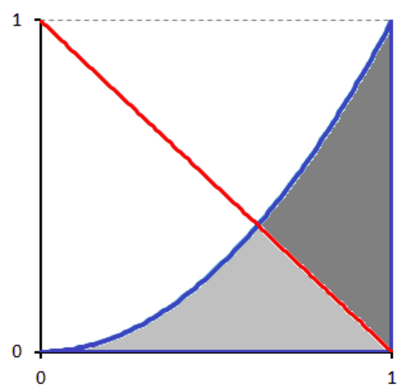
OR

$$1 - \int_{0}^{0.80}\left(\int_{0}^{x^2} 40\,x\,y^3\,dy\right)dx - \int_{0.80}^{1}\left(\int_{0}^{0.80x} 40\,x\,y^3\,dy\right)dx = \ldots$$

OR

$$1 - \int_{0}^{0.64}\left(\int_{\sqrt{y}}^{1} 40\,x\,y^3\,dx\right)dy - \int_{0.64}^{0.80}\left(\int_{1.25y}^{1} 40\,x\,y^3\,dx\right)dy = \ldots$$

c) Find $P(X + Y > 1)$.



$y = 1 - x$ and $y = x^2 \quad \Rightarrow \quad x = \dfrac{\sqrt{5}-1}{2}$

$$P(X + Y > 1) = \int_{\frac{\sqrt{5}-1}{2}}^{1}\left(\int_{1-x}^{x^2} 40\,x\,y^3\,dy\right)dx = \ldots$$

$$= \frac{509}{6} - \frac{75\sqrt{5}}{2} \approx 0.9808.$$

d) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_{0}^{x^2} 40\,x\,y^3\,dy = 10\,x^9, \qquad 0 < x < 1.$$

e) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_{\sqrt{y}}^{1} 40\,x\,y^3\,dx = 20\,y^3 - 20\,y^4 = 20\,y^3(1-y), \qquad 0 < y < 1.$$

f) Find $\mathrm{E}(X)$, $\mathrm{E}(Y)$, $\mathrm{E}(X \cdot Y)$.

$$\mathrm{E}(X) = \int_0^1 x \cdot 10\,x^9\,dx = \mathbf{\frac{10}{11}}.$$

$$\mathrm{E}(Y) = \int_0^1 y \cdot \left(20\,y^3 - 20\,y^4\right) dy = \frac{20}{5} - \frac{20}{6} = \mathbf{\frac{2}{3}}.$$

OR $$\mathrm{E}(Y) = \int_0^1 \left(\int_0^{x^2} y \cdot 40\,x\,y^3\,dy \right) dx = \int_0^1 8\,x^{11}\,dx = \frac{8}{12} = \mathbf{\frac{2}{3}}.$$

OR Y has Beta distribution with $\alpha = 4$, $\beta = 2$. $\mathrm{E}(Y) = \frac{4}{4+2} = \mathbf{\frac{2}{3}}.$

$$\mathrm{E}(XY) = \int_0^1 \left(\int_0^{x^2} x\,y \cdot 40\,x\,y^3\,dy \right) dx = \int_0^1 8\,x^{12}\,dx = \mathbf{\frac{8}{13}}.$$

Sneak preview of **4.2**:

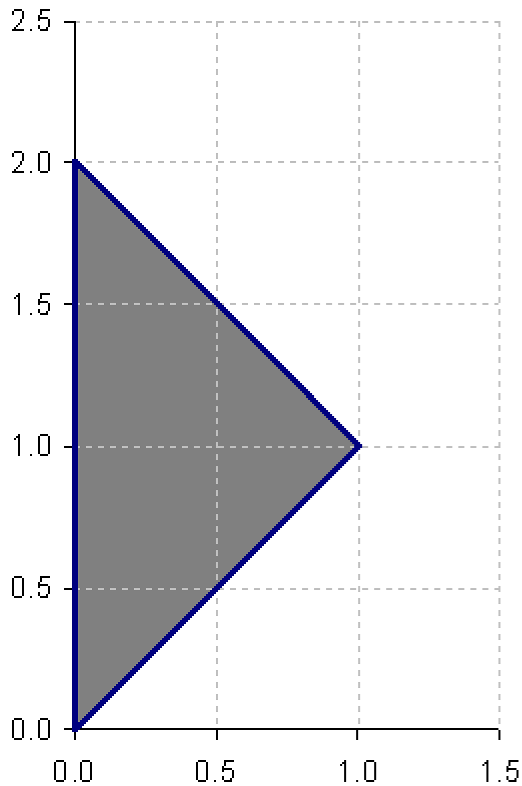
g) Find $\mathrm{Cov}(X, Y)$.

$$\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) = \frac{8}{13} - \frac{10}{11} \cdot \frac{2}{3} = \mathbf{\frac{4}{429}} \approx 0.009324.$$

2½. Suppose that the random variables X and Y have joint p.d.f. $f(x, y)$ given by

$$f(x,y) = C x^2 y, \quad 0 < x < y, \quad x + y < 2, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0 < x < y, \ x + y < 2 \}$.



b) What must the value of C be so that $f(x, y)$ is a valid joint p.d.f.?

Must have $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

$$\int_0^1 \left(\int_x^{2-x} C x^2 y\,dy \right) dx$$

$$= \int_0^1 \left(\frac{C}{2} x^2 y^2 \right) \Bigg|_{y=x}^{y=2-x} dx$$

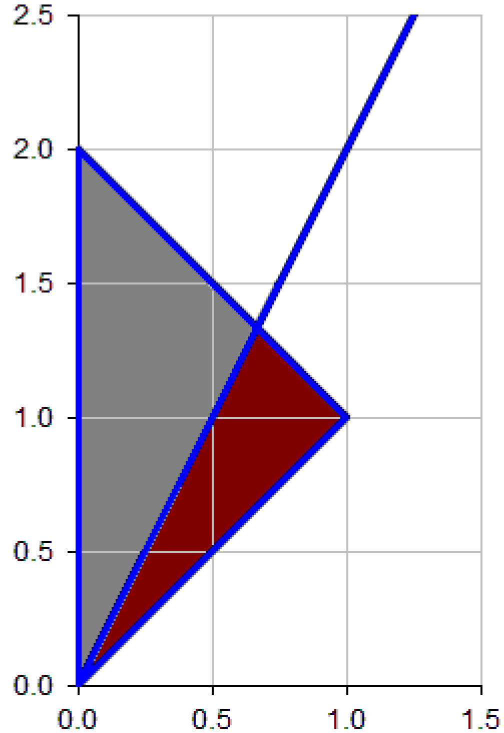
$$= \int_0^1 \left(\frac{C}{2} x^2 \left[(2-x)^2 - x^2 \right] \right) dx$$

$$= \int_0^1 \left(2 C x^2 - 2 C x^3 \right) dx$$

$$= \left(\frac{2C}{3} x^3 - \frac{C}{2} x^4 \right) \Bigg|_0^1 = \frac{C}{6} = 1.$$

$\Rightarrow \qquad C = \mathbf{6}.$

c) Find $P(Y < 2X)$.



$x + y = 2$ & $y = 2x$

$\Rightarrow \quad x = \frac{2}{3}, \; y = \frac{4}{3}.$

$$1 - \int_0^{2/3} \left(\int_{2x}^{2-x} 6x^2 y \, dy \right) dx$$

$$= 1 - \int_0^{2/3} \left(3x^2 y^2 \right) \Big|_{y=2x}^{y=2-x} dx$$

$$= 1 - \int_0^{2/3} \left(3x^2 \left[(2-x)^2 - 4x^2 \right] \right) dx$$

$$= 1 - \int_0^{2/3} \left(12x^2 - 12x^3 - 9x^4 \right) dx$$

$$= 1 - \left(4x^3 - 3x^4 - \frac{9}{5}x^5 \right) \Big|_0^{2/3} = 1 - 4\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^4 + \frac{9}{5}\left(\frac{2}{3}\right)^5 = \mathbf{\frac{87}{135}}.$$

OR

$$\int_0^{2/3} \left(\int_x^{2x} 6x^2 y \, dy \right) dx + \int_{2/3}^{1} \left(\int_x^{2-x} 6x^2 y \, dy \right) dx = \ldots$$

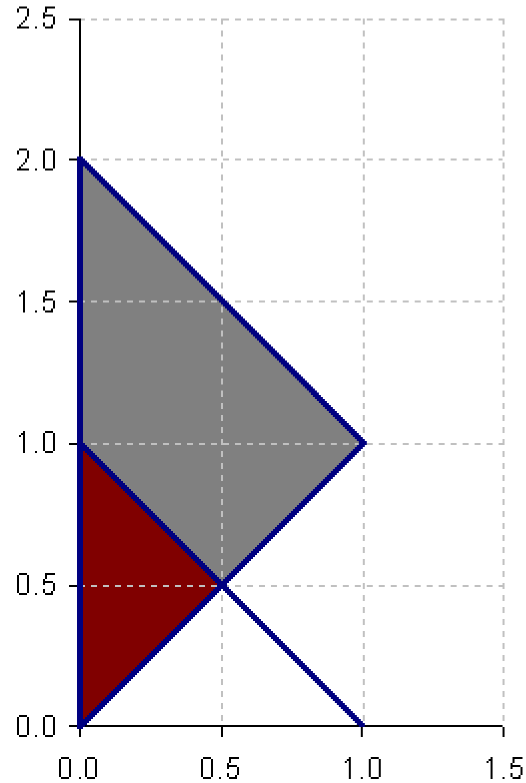
OR

$$\int_0^{1} \left(\int_{y/2}^{y} 6x^2 y \, dx \right) dy + \int_1^{4/3} \left(\int_{y/2}^{2-y} 6x^2 y \, dx \right) dy = \ldots$$

OR

$$1 - \int_0^{4/3} \left(\int_0^{y/2} 6x^2 y \, dx \right) dy - \int_{4/3}^{2} \left(\int_0^{2-y} 6x^2 y \, dx \right) dy = \ldots$$

d) Find $P(X + Y < 1)$.



$$\int_0^{0.5}\left(\int_x^{1-x} 6x^2\, y\, dy\right)dx$$

$$= \int_0^{0.5}\left(3x^2 y^2\right)\Big|_{y=x}^{y=1-x} dx$$

$$= \int_0^{0.5}\left(3x^2\left[(1-x)^2 - x^2\right]\right)dx$$

$$= \int_0^{0.5}\left(3x^2 - 6x^3\right)dx$$

$$= \left(x^3 - \frac{3}{2}x^4\right)\Big|_0^{0.5}$$

$$= \left(\frac{1}{2}\right)^3 - \frac{3}{2}\left(\frac{1}{2}\right)^4$$

$$= \frac{1}{8} - \frac{3}{32} = \mathbf{\frac{1}{32}} = \mathbf{0.03125}.$$

e) Find the marginal probability density function for X.

First, X can only take values in $(0, 1)$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^{2-x} 6x^2\, y\, dy = \left(3x^2 y^2\right)\Big|_{y=x}^{y=2-x}$$

$$= 3x^2\left\{(2-x)^2 - x^2\right\} = 12x^2 - 12x^3 = 12x^2(1-x), \quad 0 < x < 1.$$

f) Find the marginal probability density function for Y.

"Hint": Consider two cases: $0 < y < 1$ and $1 < y < 2$.

First, Y can only take values in $(0, 2)$.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \begin{cases} \int_0^y 6x^2 y\,dx & 0 < y < 1 \\ \int_0^{2-y} 6x^2 y\,dx & 1 < y < 2 \end{cases}$$

$$= \begin{cases} \left(2x^3 y\right)\Big|_{x=0}^{x=y} & 0 < y < 1 \\ \left(2x^3 y\right)\Big|_{x=0}^{x=2-y} & 1 < y < 2 \end{cases}$$

$$= \begin{cases} 2y^4 & 0 < y < 1 \\ 2y(2-y)^3 & 1 < y < 2 \end{cases}$$

g) Find $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot 12x^2(1-x)\,dx = \mathbf{0.60}.$$

h) Find $E(Y)$.

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot 2y^4\,dy + \int_1^2 y \cdot 2y(2-y)^3\,dx = \frac{1}{3} + \frac{11}{15} = \mathbf{\frac{16}{15}}.$$

i) Find $E(XY)$.

$$E(XY) = \int_0^1 \left(\int_x^{2-x} xy \cdot 6x^2 y\,dy \right) dx = \ldots = \mathbf{\frac{22}{35}}.$$