STAT 400
UIUC

Examples for 4.1, 4.4 (Part 2)

Stepanov
Dalpiaz

Independent Random Variables

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

$x \setminus y$	0	1	2	
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
	0.40	0.40	0.20	

Recall: A and B are independent if and only if $\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{A}) \cdot \mathrm{P}(\mathrm{B})$.

a) Are events $\{\mathrm{X} = 1\}$ and $\{\mathrm{Y} = 1\}$ independent?

Def Random variables X and Y are **independent** if and only if

discrete $\quad p(x, y) = p_{\mathrm{X}}(x) \cdot p_{\mathrm{Y}}(y) \quad$ for all x, y.

continuous $\quad f(x, y) = f_{\mathrm{X}}(x) \cdot f_{\mathrm{Y}}(y) \quad$ for all x, y.

$\mathrm{F}(x, y) = \mathrm{P}(\mathrm{X} \le x, \mathrm{Y} \le y)$. $\qquad f(x, y) = \partial^2 \mathrm{F}(x, y) / \partial x \, \partial y$.

Def Random variables X and Y are **independent** if and only if

$$\mathrm{F}(x, y) = \mathrm{F}_{\mathrm{X}}(x) \cdot \mathrm{F}_{\mathrm{Y}}(y) \quad \text{for all } x, y.$$

b) Are random variables X and Y independent?

2. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 60\,x^2\,y & 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Recall:

$$f_X(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1,$$

$$f_Y(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1.$$

Are random variables X and Y independent?

3. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} x + y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

4. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 12\,x\,(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

If random variables X and Y are independent, then

$$\mathrm{E}(g(\mathrm{X}) \cdot h(\mathrm{Y})) = \mathrm{E}(g(\mathrm{X})) \cdot \mathrm{E}(h(\mathrm{Y})).$$

5. Suppose the probability density functions of T_1 and T_2 are

$$f_{\mathrm{T}_1}(x) = \alpha\, e^{-\alpha x}, \quad x > 0, \qquad f_{\mathrm{T}_2}(y) = \beta\, e^{-\beta y}, \quad y > 0,$$

respectively. Suppose T_1 and T_2 are independent. Find $\mathrm{P}(2\,\mathrm{T}_1 > \mathrm{T}_2)$.

6. Let X and Y be two independent random variables, X has a Geometric distribution with the probability of "success" $p = {}^1\!/_3$, Y has a Poisson distribution with mean 3. That is,

$$p_{\mathrm{X}}(x) = \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{x-1}, \qquad x = 1, 2, 3, \ldots,$$

$$p_{\mathrm{Y}}(y) = \frac{3^y e^{-3}}{y!}, \quad y = 0, 1, 2, 3, \ldots .$$

a) Find $\mathrm{P}(\mathrm{X} = \mathrm{Y})$.

b) Find $\mathrm{P}(\mathrm{X} = 2\,\mathrm{Y})$.