Independent Random Variables

1. Consider the following joint probability distribution \( p(x, y) \) of two random variables \( X \) and \( Y \):

\[
\begin{array}{c|ccc}
 x & 0 & 1 & 2 \\
1 & 0.15 & 0.10 & 0 & 0.25 \\
2 & 0.25 & 0.30 & 0.20 & 0.75 \\
\end{array}
\]

Recall: \( A \) and \( B \) are independent if and only if \( P(A \cap B) = P(A) \cdot P(B) \).

a) Are events \( \{X = 1\} \) and \( \{Y = 1\} \) independent?

\begin{equation}
\text{Def} \quad \text{Random variables } X \text{ and } Y \text{ are independent if and only if}
\end{equation}

\begin{align*}
\text{discrete} & \quad p(x, y) = p_X(x) \cdot p_Y(y) \quad \text{for all } x, y. \\
\text{continuous} & \quad f(x, y) = f_X(x) \cdot f_Y(y) \quad \text{for all } x, y.
\end{align*}

\begin{align*}
F(x, y) &= P(X \leq x, Y \leq y). \\
f(x, y) &= \frac{\partial^2 F(x, y)}{\partial x \partial y}.
\end{align*}

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\text{Def} \quad \text{Random variables } X \text{ and } Y \text{ are independent if and only if}
\end{equation}

\[
F(x, y) = F_X(x) \cdot F_Y(y) \quad \text{for all } x, y.
\]

b) Are random variables \( X \) and \( Y \) independent?
2. Let the joint probability density function for \((X, Y)\) be
\[
f(x, y) = \begin{cases} 
60x^2 y & 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ x + y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]
Recall: 
\[
f_X(x) = 30x^2 (1 - x)^2, \quad 0 < x < 1, \\
f_Y(y) = 20y(1 - y)^3, \quad 0 < y < 1.
\]
Are random variables \(X\) and \(Y\) independent?

3. Let the joint probability density function for \((X, Y)\) be
\[
f(x, y) = \begin{cases} 
x + y & 0 \leq x \leq 1, \ 0 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]
Are \(X\) and \(Y\) independent?

4. Let the joint probability density function for \((X, Y)\) be
\[
f(x, y) = \begin{cases} 
12x (1 - x) e^{-2y} & 0 \leq x \leq 1, \ y \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]
Are \(X\) and \(Y\) independent?
If random variables \( X \) and \( Y \) are independent, then
\[
E(g(X) \cdot h(Y)) = E(g(X)) \cdot E(h(Y)).
\]

5. Suppose the probability density functions of \( T_1 \) and \( T_2 \) are
\[
f_{T_1}(x) = \alpha e^{-\alpha x}, \quad x > 0, \quad f_{T_2}(y) = \beta e^{-\beta y}, \quad y > 0,
\]
respectively. Suppose \( T_1 \) and \( T_2 \) are independent. Find \( P(2T_1 > T_2) \).

6. Let \( X \) and \( Y \) be two independent random variables, \( X \) has a Geometric distribution with the probability of “success” \( p = \frac{1}{3} \), \( Y \) has a Poisson distribution with mean 3. That is,
\[
p_{X}(x) = \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^{x-1}, \quad x = 1, 2, 3, \ldots ,
\]
\[
p_{Y}(y) = \frac{3^y e^{-3}}{y!}, \quad y = 0, 1, 2, 3, \ldots .
\]
a) Find \( P(X = Y) \).

b) Find \( P(X = 2Y) \).