STAT 400
UIUC

Examples for 4.1, 4.4 (3)

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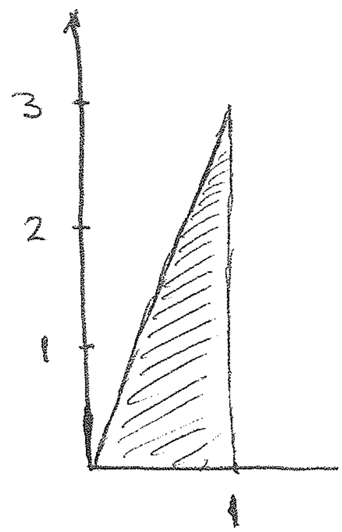
2¾. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = C\,x^3\,y, \qquad 0 < x < 1, \quad 0 < y < 3x, \qquad \text{zero otherwise.}$$

a) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.



Must have

$$1 = \int_0^1 \left(\int_0^{3x} C\,x^3 y \, dy \right) dx$$

$$= \int_0^1 \frac{C}{2} x^3 (3x)^2 \, dx$$

$$= \frac{9C}{2} \int_0^1 x^5 \, dx = \frac{3C}{4}.$$

$$\Rightarrow \quad C = \frac{4}{3}.$$

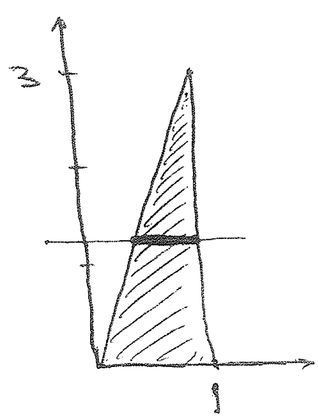
b) Find the marginal probability density function of X, $f_X(x)$.

Be sure to include its support.

$$f_X(x) = \int_0^{3x} \frac{4}{3} x^3 y \, dy = 6x^5, \qquad 0 < x < 1.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.

Be sure to include its support.

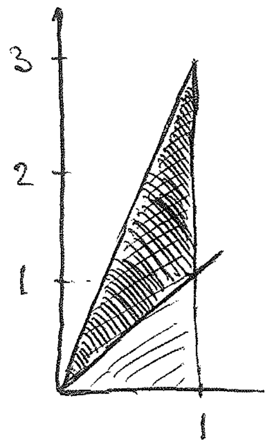


$$f_Y(y) = \int_{y/3}^{1} \frac{4}{3} x^3 y \, dx$$

$$= \left(\frac{1}{3} x^4 y \right)_{x = \frac{y}{3}}^{x=1}$$

$$= \frac{1}{3} y - \frac{1}{243} y^5, \qquad 0 < y < 3.$$

d) Find $P(X < Y)$.



$$\int_0^1 \left(\int_x^{3x} \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$1 - \int_0^1 \left(\int_0^x \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$1 - \int_0^1 \left(\int_y^1 \frac{4}{3} x^3 y \, dx \right) dy$$

OR

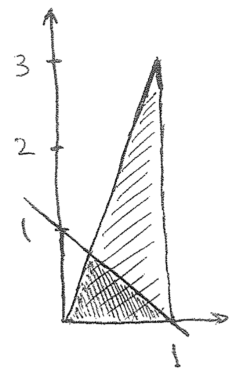
$$\int_0^1 \left(\int_{y/3}^{y} \frac{4}{3} x^3 y \, dx \right) dy + \int_1^3 \left(\int_{y/3}^{1} \frac{4}{3} x^3 y \, dx \right) dy$$

$$\int_0^1 \left(\int_x^{3x} \frac{4}{3} x^3 y \, dy \right) dx = \int_0^1 \frac{16}{3} x^5 \, dx = \mathbf{\frac{8}{9}}.$$

$$1 - \int_0^1 \left(\int_0^x \frac{4}{3} x^3 y \, dy \right) dx = 1 - \int_0^1 \frac{2}{3} x^5 \, dx = 1 - \frac{1}{9} = \mathbf{\frac{8}{9}}.$$

$$1 - \int_0^1 \left(\int_y^1 \frac{4}{3} x^3 y \, dx \right) dy = 1 - \int_0^1 \frac{1}{3} \left(y - y^5 \right) dy = 1 - \frac{1}{6} + \frac{1}{18} = \mathbf{\frac{8}{9}}.$$

e) Find $P(X+Y<1)$.



$$\int_0^{3/4}\left(\int_{y/3}^{1-y}\frac{4}{3}x^3 y\,dx\right)dy$$

OR

$$1-\int_{1/4}^{1}\left(\int_{1-x}^{3x}\frac{4}{3}x^3 y\,dy\right)dx$$

OR

$$\int_0^{1/4}\left(\int_0^{3x}\frac{4}{3}x^3 y\,dy\right)dx+\int_{1/4}^{1}\left(\int_0^{1-x}\frac{4}{3}x^3 y\,dy\right)dx$$

$y=3x$

$y=1-x$

$\Rightarrow x=\frac{1}{4}$

$y=\frac{3}{4}$

$$1-\int_{1/4}^{1}\left(\int_{1-x}^{3x}\frac{4}{3}x^3 y\,dy\right)dx = 1-\int_{1/4}^{1}\frac{2}{3}x^3\left(8x^2+2x-1\right)dx$$

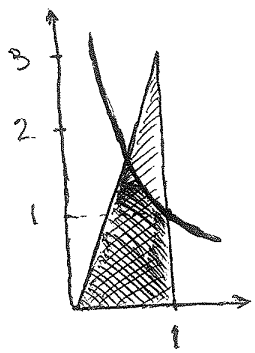
$$= 1-\left(\frac{8}{9}x^6+\frac{4}{15}x^5-\frac{1}{6}x^4\right)\Bigg|_{1/4}^{1} = \mathbf{\frac{7}{640}} = \mathbf{0.0109375}.$$

$$\int_0^{1/4}\left(\int_0^{3x}\frac{4}{3}x^3 y\,dy\right)dx + \int_{1/4}^{1}\left(\int_0^{1-x}\frac{4}{3}x^3 y\,dy\right)dx$$

$$= \int_0^{1/4} 6x^5\,dx + \int_{1/4}^{1}\frac{2}{3}x^3\left(1-2x+x^2\right)dx$$

$$= \frac{1}{4096} + \left(\frac{1}{6}x^4-\frac{4}{15}x^5+\frac{1}{9}x^6\right)\Bigg|_{1/4}^{1} = \mathbf{\frac{7}{640}} = \mathbf{0.0109375}.$$

f) Find $P(X \cdot Y < 1)$.



$$1 - \int_{1/\sqrt{3}}^{1} \left(\int_{1/x}^{3x} \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$\int_{0}^{1/\sqrt{3}} \left(\int_{0}^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{1/\sqrt{3}}^{1} \left(\int_{0}^{1/x} \frac{4}{3} x^3 y \, dy \right) dx$$

OR

$$\int_{0}^{1} \left(\int_{y/3}^{1} \frac{4}{3} x^3 y \, dx \right) dy + \int_{1}^{\sqrt{3}} \left(\int_{y/3}^{1/y} \frac{4}{3} x^3 y \, dx \right) dy$$

$y = 3x$

$x \cdot y = 1$

$\Rightarrow \quad x = \frac{1}{\sqrt{3}}$

$y = \sqrt{3}$

$$1 - \int_{1/\sqrt{3}}^{1} \left(\int_{1/x}^{3x} \frac{4}{3} x^3 y \, dy \right) dx = 1 - \int_{1/\sqrt{3}}^{1} \frac{2}{3} x^3 \left(9x^2 - \frac{1}{x^2} \right) dx$$

$$= 1 - \int_{1/\sqrt{3}}^{1} \left(6x^5 - \frac{2x}{3} \right) dx = 1 - \left(x^6 - \frac{x^2}{3} \right) \Bigg|_{1/\sqrt{3}}^{1}$$

$$= \mathbf{\frac{7}{27}} \approx 0.25926.$$

$$\int_{0}^{1/\sqrt{3}} \left(\int_{0}^{3x} \frac{4}{3} x^3 y \, dy \right) dx + \int_{1/\sqrt{3}}^{1} \left(\int_{0}^{1/x} \frac{4}{3} x^3 y \, dy \right) dx$$

$$= \int_{0}^{1/\sqrt{3}} 6x^5 \, dx + \int_{1/\sqrt{3}}^{1} \frac{2x}{3} \, dx = \frac{1}{27} + \left(\frac{1}{3} - \frac{1}{9} \right) = \mathbf{\frac{7}{27}} \approx 0.25926.$$

g) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

$f_{X,Y}(x, y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

$$\text{E}(X) = \int_0^1 x \cdot 6x^5 \, dx = \frac{6}{7}.$$

$$\text{E}(Y) = \int_0^3 y \cdot \left(\frac{1}{3} y - \frac{1}{243} y^5 \right) dy = \left(\frac{1}{9} y^3 - \frac{1}{1701} y^7 \right) \Bigg|_0^3 = 3 - \frac{9}{7} = \frac{12}{7}.$$

$$\text{E}(XY) = \int_0^1 \left(\int_0^{3x} x y \cdot \frac{4}{3} x^3 y \, dy \right) dx = \int_0^1 \left(\int_0^{3x} \frac{4}{3} x^4 y^2 \, dy \right) dx = \int_0^1 12 x^7 \, dx = \frac{3}{2}.$$

$$\text{Cov}(X, Y) = \text{E}(XY) - \text{E}(X) \cdot \text{E}(Y) = \frac{3}{2} - \frac{6}{7} \times \frac{12}{7} = \mathbf{\frac{3}{98}}.$$