STAT 400
UIUC

Examples for 4.2

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4.2 Covariance and Correlation Coefficient

Covariance of X and Y

$$\sigma_{XY} = \operatorname{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$

(a) $\operatorname{Cov}(X, X) = \operatorname{Var}(X)$;

(b) $\operatorname{Cov}(X, Y) = \operatorname{Cov}(Y, X)$;

(c) $\operatorname{Cov}(aX + b, Y) = a\operatorname{Cov}(X, Y)$;

(d) $\operatorname{Cov}(X + Y, W) = \operatorname{Cov}(X, W) + \operatorname{Cov}(Y, W)$.

$$\operatorname{Cov}(aX + bY, cX + dY) = ac\operatorname{Var}(X) + (ad + bc)\operatorname{Cov}(X, Y) + bd\operatorname{Var}(Y).$$

$$\operatorname{Var}(aX + bY) = \operatorname{Cov}(aX + bY, aX + bY) = a^2\operatorname{Var}(X) + 2ab\operatorname{Cov}(X, Y) + b^2\operatorname{Var}(Y).$$

0. Find in terms of σ_X^2, σ_Y^2, and σ_{XY}:

a) $\operatorname{Cov}(2X + 3Y, X - 2Y)$,

b) $\operatorname{Var}(2X + 3Y)$,

c) $\operatorname{Var}(X - 2Y)$.

Correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X,\ Y)}{\sqrt{\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}} = \text{E}\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]$$

(a) $-1 \le \rho_{XY} \le 1$;

(b) ρ_{XY} is either $+1$ or -1 if and only if X and Y are linear functions of one another.

If random variables X and Y are independent, then

$$\text{E}(g(X) \cdot h(Y)) = \text{E}(g(X)) \cdot \text{E}(h(Y)).$$

$\Rightarrow$ $\quad \text{Cov}(X, Y) = \sigma_{XY} = 0, \qquad \text{Corr}(X, Y) = \rho_{XY} = 0.$

1. Consider the following joint probability distribution $p(x, y)$ of two random variables X and Y:

	y			
x	0	1	2	$p_X(x)$
1	0.15	0.10	0	0.25
2	0.25	0.30	0.20	0.75
$p_Y(y)$	0.40	0.40	0.20	1.00

Recall:

$\text{E}(X) = 1.75$,

$\text{E}(Y) = 0.8$,

$\text{E}(XY) = 1.5$.

Find $\text{Cov}(X, Y) = \sigma_{XY}$ and $\text{Corr}(X, Y) = \rho_{XY}$.

2. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} 60\,x^2\,y & 0\le x\le 1,\ 0\le y\le 1,\ x+y\le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\operatorname{Cov}(X,Y) = \sigma_{XY}$ and $\operatorname{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = 30\,x^2\,(1-x)^2, \quad 0 < x < 1,$ $\quad \mathrm{E}(X) = \dfrac{1}{2},$

$f_Y(y) = 20\,y\,(1-y)^3, \quad 0 < y < 1,$ $\quad \mathrm{E}(Y) = \dfrac{1}{3},$ $\quad \mathrm{E}(XY) = \dfrac{1}{7}.$

3. Let the joint probability density function for (X, Y) be

$$f(x,y)=\begin{cases} x+y & 0\le x\le 1, 0\le y\le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\operatorname{Cov}(X,Y) = \sigma_{XY}$ and $\operatorname{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = x+\dfrac{1}{2}, \quad 0 < x < 1.$ $\quad f_Y(y) = y+\dfrac{1}{2}, \quad 0 < y < 1.$

4. Let the joint probability density function for (X, Y) be

$$f(x,y) = \begin{cases} 12\,x(1-x)\,e^{-2y} & 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\operatorname{Cov}(X,Y) = \sigma_{XY}$ and $\operatorname{Corr}(X,Y) = \rho_{XY}$.

Recall: $f_X(x) = 6\,x(1-x),\ \ 0 < x < 1.$ $f_Y(y) = 2\,e^{-2y},\ \ y > 0.$