STAT 400
UIUC

Answers for 5.7

Stepanov
Dalpiaz

Normal Approximation to Binomial Distribution:

	Normal	Binomial
mean	μ	$n \times p$
standard deviation	σ	$\sqrt{n \times p \times (1 - p)}$

1. Binomial distribution, $n = 25$, $p = 0.50$.

Normal approximation:

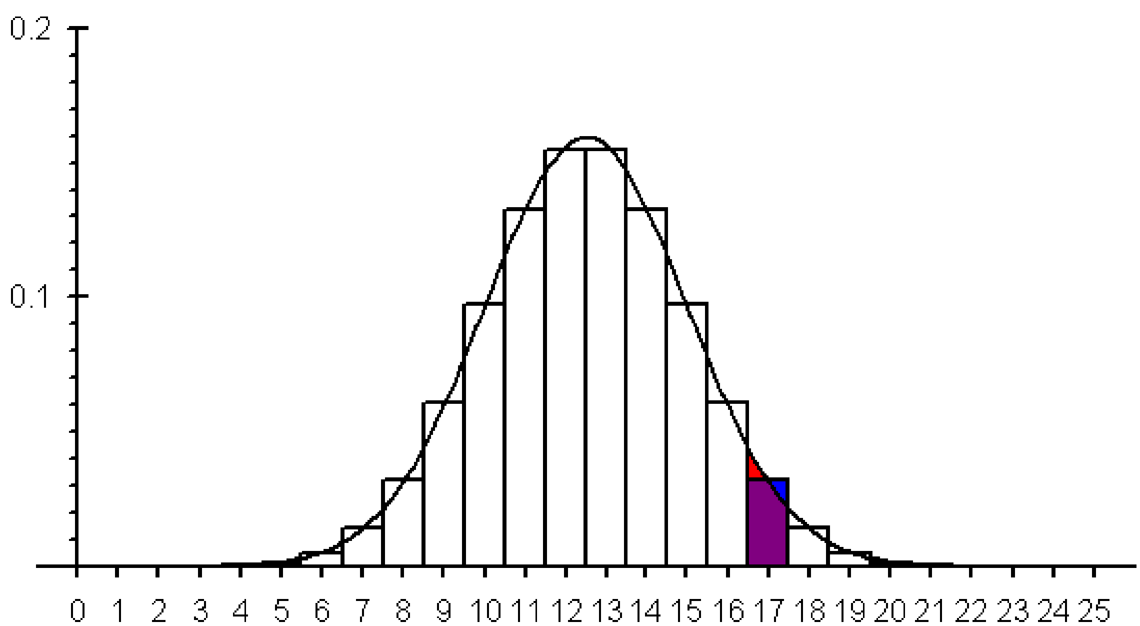
mean = $n \times p = 25 \times 0.50 = 12.5$.

$n \times p \times (1 - p) = 25 \times 0.50 \times 0.50 = 6.25$. SD = $\sqrt{6.25} = 2.5$.

a) P(X = 17) = PMF @ 17 = **0.0322**.

b) $P(X = 17) = P(16.5 \le X \le 17.5) \approx P\left(\frac{16.5 - 12.5}{2.5} \le Z \le \frac{17.5 - 12.5}{2.5}\right)$

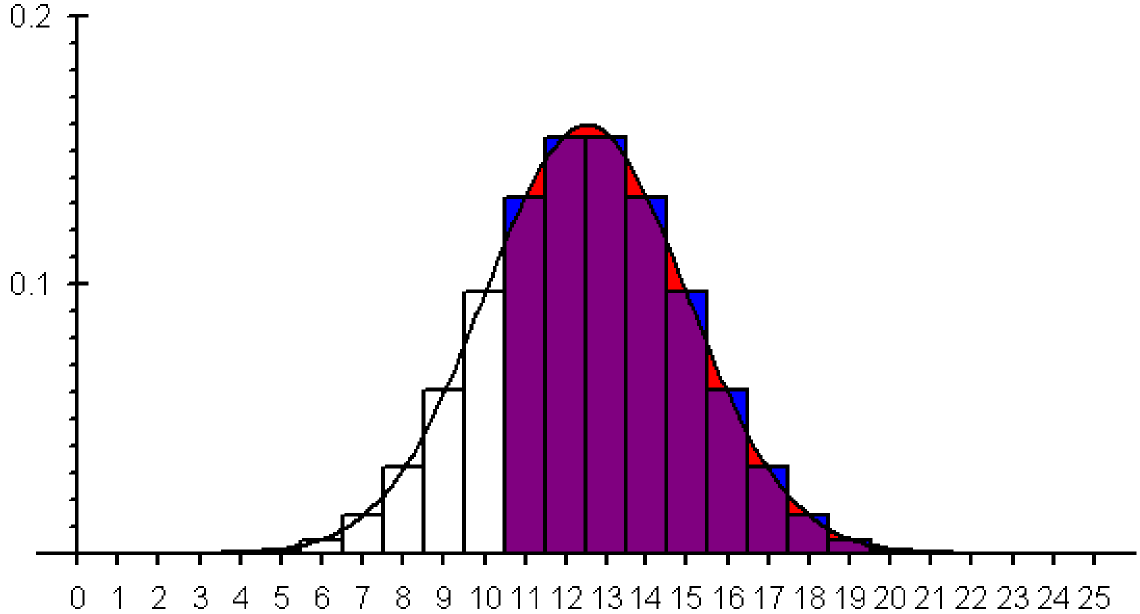
$= P(1.60 \le Z \le 2.00) = 0.9772 - 0.9452 =$ **0.0320**.



c) $P(X \geq 11) = 1 - \text{CDF} @ 10 = 1 - 0.2122 = \mathbf{0.7878}$.

d) $P(X \geq 11) = P(X \geq 10.5) \approx P\left(Z \geq \dfrac{10.5 - 12.5}{2.5}\right)$

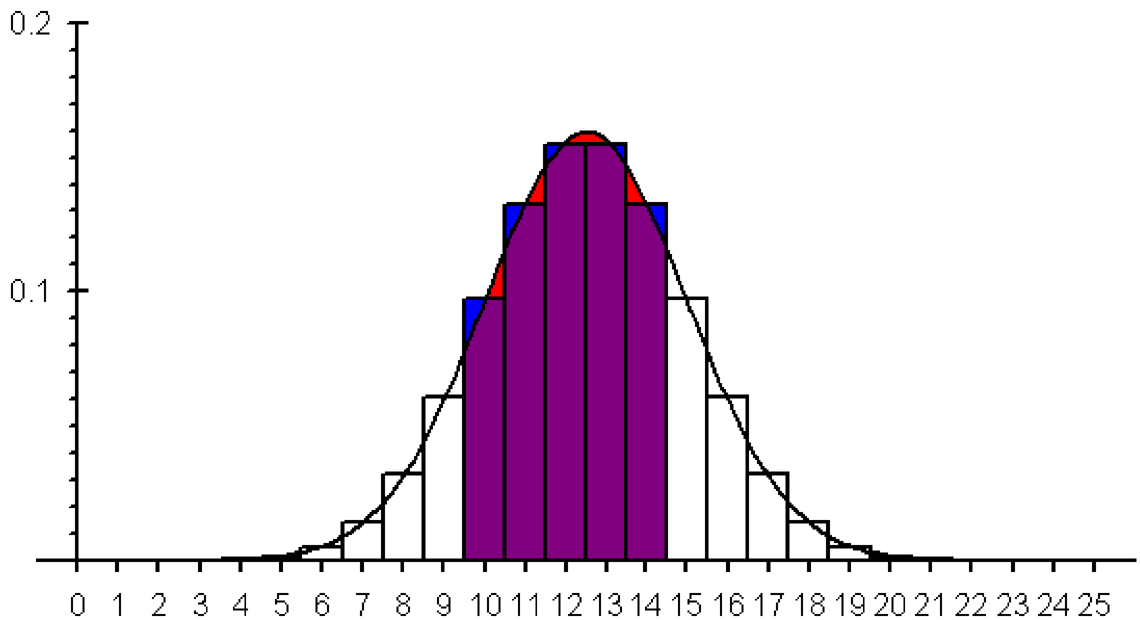
$= P(Z \geq -0.80) = 1 - 0.2119 = \mathbf{0.7881}$.



e) $P(10 \leq X \leq 14) = \text{CDF} @ 14 - \text{CDF} @ 9 = 0.7878 - 0.1148 = \mathbf{0.6730}$.

f) $P(10 \leq X \leq 14) = P(9.5 \leq X \leq 14.5) \approx P\left(\dfrac{9.5 - 12.5}{2.5} \leq Z \leq \dfrac{14.5 - 12.5}{2.5}\right)$

$= P(-1.20 \leq Z \leq 0.80) = 0.7881 - 0.1151 = \mathbf{0.6730}$.



2. Let X = number of passengers who do not cancel their reservations.

Then X has Binomial distribution, $n = 100$, $p = 0.85$.

Normal approximation:

$$\mu = 100 \times 0.85 = 85, \quad \sigma^2 = 100 \times 0.85 \times 0.15 = 12.75. \quad \sigma = 3.57.$$

$$P(X \le 92) = P(X \le 92.5) \approx P\left(Z \le \frac{92.5 - 85}{3.57}\right) = P(Z \le 2.10) = \mathbf{0.9821}.$$

Binomial: $P(X \le 92) = 0.9878.$

2.5. A fair 6-sided die is rolled 180 times. The sum of the outcomes is likely to be around __________, give or take __________ or so.

The average of the outcomes is likely to be around __________, give or take __________ or so.

$$\mu = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5.$$

$$\sigma = \sqrt{\frac{(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2}{6}}$$

$$= \sqrt{\frac{6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25}{6}} = \sqrt{\frac{17.5}{6}} \approx 1.708.$$

$$E(\text{Sum}) = n \times \mu = 180 \times 3.5 = 630.$$

$$SD(\text{Sum}) = \sqrt{n} \times \sigma \approx \sqrt{180} \times 1.708 \approx 22.9.$$

The sum of the outcomes is likely to be around **630**, give or take **23** or so.

$$E(\text{Average}) = \mu = 3.5.$$

$$SD(\text{Average}) = \sigma / \sqrt{n} \approx 1.708 / \sqrt{180} \approx 0.1273.$$

The average of the outcomes is likely to be around **3.5**, give or take **0.13** or so.

A fair 6-sided die is rolled 180 times. The number of 6's is likely to be around __________, give or take __________ or so.

Let X denote the number of 6's.

Binomial distribution, $n = 180$, $p = {}^{1}\!/_{6}$.

$\mathrm{E}(\mathrm{X}) = n \times p = 180 \times {}^{1}\!/_{6} = 30.$

$\mathrm{SD}(\mathrm{X}) = \sqrt{n \times p \times (1-p)} = \sqrt{180 \times {}^{1}\!/_{6} \times {}^{5}\!/_{6}} = 5.$

The number of 6's is likely to be around **30**, give or take **5** or so.

3. Binomial distribution, $n = 180$, $p = {}^{1}\!/_{6}$.

a) $\mathrm{P}(\mathrm{X} = 35) = {}_{180}\mathrm{C}_{35} \cdot \left(\frac{1}{6}\right)^{35} \cdot \left(\frac{5}{6}\right)^{145} = \mathbf{0.0464}.$

Normal approximation:

mean $= n \times p = 180 \times {}^{1}\!/_{6} = 30.$

$n \times p \times (1-p) = 180 \times {}^{1}\!/_{6} \times {}^{5}\!/_{6} = 25.$ SD $= \sqrt{25} = 5.$

b) $\mathrm{P}(\mathrm{X} = 35) = \mathrm{P}(34.5 < \mathrm{X} < 35.5) \approx \mathrm{P}(0.90 < \mathrm{Z} < 1.10)$

$= 0.8643 - 0.8159 = \mathbf{0.0484}.$

c) $\mathrm{P}(\mathrm{X} \ge 35) = \mathrm{P}(\mathrm{X} > 34.5) \approx \mathrm{P}(\mathrm{Z} > 0.90) = 1 - 0.8159 = \mathbf{0.1841}.$

Binomial: $\mathrm{P}(\mathrm{X} \ge 35) = \sum_{k=35}^{180} {}_{180}\mathrm{C}_{k} \cdot \left(\frac{1}{6}\right)^{k} \cdot \left(\frac{5}{6}\right)^{180-k} \approx 0.18283.$

d) $\mathrm{P}(20 \le \mathrm{X} \le 40) = \mathrm{P}(19.5 < \mathrm{X} < 40.5) \approx \mathrm{P}(-2.10 < \mathrm{Z} < 2.10) = \mathbf{0.9642}.$

Binomial: $\mathrm{P}(20 \le \mathrm{X} \le 40) = \sum_{k=20}^{40} {}_{180}\mathrm{C}_{k} \cdot \left(\frac{1}{6}\right)^{k} \cdot \left(\frac{5}{6}\right)^{180-k} \approx 0.965.$

e)* Use Normal approximation to find the probability that the sum of the results is between 600 and 640 (both inclusive)? [Recall: $\mu = 3.5$, $\sigma^2 = {}^{17.5}/_6 = {}^{35}/_{12}$.]

$P(600 \le \text{Sum} \le 640) = P(599.5 < \text{Sum} < 640.5)$

$$\approx P\left(\frac{599.5 - 180 \cdot 3.5}{\sqrt{180} \cdot \sqrt{35/12}} < Z < \frac{640.5 - 180 \cdot 3.5}{\sqrt{180} \cdot \sqrt{35/12}} \right)$$

$= P(-1.33 < Z < 0.46) = \mathbf{0.5854}$.

4. Poisson distribution, $\lambda = 1.4 \cdot 52 = 72.8$.

a) $P(X = 68) = \dfrac{72.8^{68} \cdot e^{-72.8}}{68!} = \mathbf{0.0411}$.

Normal approximation:

$\mu = \lambda = 72.8$. $\sigma = \sqrt{\lambda} = \sqrt{72.8} = 8.5323$.

b) $P(X = 68) = P(67.5 < X < 68.5) \approx P(-0.62 < Z < -0.50)$

$= 0.3085 - 0.2676 = \mathbf{0.0409}$.

c) $P(X \le 70) = P(X < 70.5) \approx P(Z < -0.27) = \mathbf{0.3936}$.

Poisson: $P(X \le 70) = \displaystyle\sum_{k=0}^{70} \frac{72.8^k \cdot e^{-72.8}}{k!} \approx 0.40078$.

d) $P(65 \le X \le 80) = P(64.5 < X < 80.5) \approx P(-0.97 < Z < 0.90)$

$= 0.8159 - 0.1660 = \mathbf{0.6499}$.

Poisson: $P(65 \le X \le 80) = \displaystyle\sum_{k=65}^{80} \frac{72.8^k \cdot e^{-72.8}}{k!} \approx 0.65218$.

"Hype" for 0.5 continuity correction:

Let X be a Binomial$(\,n = 400,\ p = 0.80\,)$ random variable.

$P(\,X \le 312\,)$ = **0.1738**.

	A	B
1	=BINOMDIST(312,400,0.8,1)	
2		

$\Rightarrow$

	A	B
1	0.173821	
2		

Without 0.5 continuity correction:

$P(\,X \le 312\,)$

$$z = \frac{312 - 400 \cdot 0.80}{\sqrt{400 \cdot 0.80 \cdot 0.20}} = -1.00$$

$P(\,Z < -1.00\,)$ = 0.1587.

☹

With 0.5 continuity correction:

$P(\,X \le 312\,) = P(\,X \le 312.5\,)$

$$z = \frac{312.5 - 400 \cdot 0.80}{\sqrt{400 \cdot 0.80 \cdot 0.20}} = -0.9375$$

$P(\,Z < -0.94\,)$ = 0.1736.

$P(\,Z < -0.9375\,)$ = 0.17425.

☺