p.m.f. or p.d.f. \( f(x; \theta), \quad \theta \in \Omega. \quad \Omega \) – parameter space.

1. Suppose \( \Omega = \{1, 2, 3\} \) and the p.d.f. \( f(x; \theta) \) is

\[
\begin{align*}
\theta = 1: & \quad f(1; 1) = 0.6, \quad f(2; 1) = 0.1, \quad f(3; 1) = 0.1, \quad f(4; 1) = 0.2. \\
\theta = 2: & \quad f(1; 2) = 0.2, \quad f(2; 2) = 0.3, \quad f(3; 2) = 0.3, \quad f(4; 2) = 0.2. \\
\theta = 3: & \quad f(1; 3) = 0.3, \quad f(2; 3) = 0.4, \quad f(3; 3) = 0.2, \quad f(4; 3) = 0.1.
\end{align*}
\]

What is the maximum likelihood estimate of \( \theta \) (based on only one observation of \( X \)) if …

a) \( X = 1; \)

b) \( X = 2; \)

c) \( X = 3; \)

d) \( X = 4. \)

Likelihood function:

\[
L(\theta) = L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta) = f(x_1; \theta) \cdot \ldots \cdot f(x_n; \theta)
\]

It is often easier to consider

\[
\ln L(\theta) = \sum_{i=1}^{n} \ln f(x_i; \theta).
\]

Maximum Likelihood Estimator:

\[
\hat{\theta} = \arg \max L(\theta) = \arg \max \ln L(\theta).
\]

Method of Moments:

\[
E(X) = g(\theta). \quad \text{Set} \quad \bar{x} = g(\tilde{\theta}). \quad \text{Solve for } \tilde{\theta}.
\]
2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a Poisson distribution with mean $\lambda$, $\lambda > 0$. That is,
\[ P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, 3, \ldots. \]

a) Obtain the method of moments estimator of $\lambda$, $\tilde{\lambda}$.

b) Obtain the maximum likelihood estimator of $\lambda$, $\hat{\lambda}$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a Geometric distribution with probability of “success” $p$, $0 < p < 1$. That is,
\[ P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \ldots. \]

a) Obtain the method of moments estimator of $p$, $\tilde{p}$.

b) Obtain the maximum likelihood estimator of $p$, $\hat{p}$.
4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from the distribution with probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} \cdot x & 0 \leq x \leq 1 \\ \frac{1-\theta}{\theta} & \text{otherwise} \\ 0 < \theta < \infty. \end{cases}$$

a) Obtain the method of moments estimator of $\theta$, $\tilde{\theta}$.

Method of Moments:

$$E(X) = g(\theta).$$

Set $\bar{X} = g(\tilde{\theta}).$ Solve for $\tilde{\theta}$.

b) Obtain the maximum likelihood estimator of $\theta$, $\hat{\theta}$.

Likelihood function:

$$L(\theta) = L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta) = f(x_1; \theta) \cdot \ldots \cdot f(x_n; \theta)$$

Maximum Likelihood Estimator: $\hat{\theta} = \arg \max L(\theta) = \arg \max \ln L(\theta)$. 

4. (continued)
c) Suppose \( n = 3 \), and \( x_1 = 0.2, \ x_2 = 0.3, \ x_3 = 0.5 \). Compute the values of the method of moments estimate and the maximum likelihood estimate for \( \theta \).

5. Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from \( N(\theta_1, \theta_2) \), where \( \Omega = \{ (\theta_1, \theta_2) : -\infty < \theta_1 < \infty, \ 0 < \theta_2 < \infty \} \). That is, here we let \( \theta_1 = \mu \) and \( \theta_2 = \sigma^2 \).

a) Obtain the maximum likelihood estimator of \( \theta_1, \hat{\theta}_1 \), and of \( \theta_2, \hat{\theta}_2 \).

b) Obtain the method of moments estimator of \( \theta_1, \tilde{\theta}_1 \), and of \( \theta_2, \tilde{\theta}_2 \).