STAT 400
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Answers for 6.4, Part 1

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p.m.f. or p.d.f. $f(x;\theta)$, $\theta \in \Omega$. Ω – parameter space.

1. Suppose $\Omega = \{1, 2, 3\}$ and the p.m.f. $f(x;\theta)$ is

$\theta = 1$: $f(1;1) = 0.6$, $f(2;1) = 0.1$, $f(3;1) = 0.1$, $f(4;1) = 0.2$.

$\theta = 2$: $f(1;2) = 0.2$, $f(2;2) = 0.3$, $f(3;2) = 0.3$, $f(4;2) = 0.2$.

$\theta = 3$: $f(1;3) = 0.3$, $f(2;3) = 0.4$, $f(3;3) = 0.2$, $f(4;3) = 0.1$.

What is the maximum likelihood estimate of θ (based on only one observation of X) if …

a) X = 1;

$f(1;1) = 0.6 \Leftarrow$
$f(1;2) = 0.2$ $\Rightarrow$ $\hat{\theta} =$ **1**.
$f(1;3) = 0.3$

b) X = 2;

$f(2;1) = 0.1$
$f(2;2) = 0.3$ $\Rightarrow$ $\hat{\theta} =$ **3**.
$f(2;3) = 0.4 \Leftarrow$

c) X = 3;

$f(3;1) = 0.1$
$f(3;2) = 0.3 \Leftarrow$ $\Rightarrow$ $\hat{\theta} =$ **2**.
$f(3;3) = 0.2$

d) X = 4.

$f(4;1) = 0.2 \Leftarrow$
$f(4;2) = 0.2 \Leftarrow$ $\Rightarrow$ $\hat{\theta} =$ **1 or 2**.
$f(4;3) = 0.1$ (maximum likelihood estimate may not be unique)

Likelihood function:

$$L(\theta) = L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta) = f(x_1; \theta) \cdot \ldots \cdot f(x_n; \theta)$$

It is often easier to consider $\ln L(\theta) = \sum_{i=1}^{n} \ln f(x_i; \theta)$.

Maximum Likelihood Estimator: $\hat{\theta} = \arg\max L(\theta) = \arg\max \ln L(\theta)$.

Method of Moments:

$E(X) = g(\theta)$. Set $\overline{X} = g(\tilde{\theta})$. Solve for $\tilde{\theta}$.

0. Consider a single observation X of a Binomial random variable with n trials and probability of "success" p. That is,

$$P(X = k) = {}_nC_k \, p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

a) Obtain the method of moments estimator of p, $\tilde{p}$.

Binomial: $E(X) = n\,p$

$$X = n\,\tilde{p} \qquad \Rightarrow \qquad \tilde{p} = \frac{X}{n}.$$

b) Obtain the maximum likelihood estimator of p, $\hat{p}$.

$$L(p) = {}_nC_X \, p^X (1-p)^{n-X}$$

$$\ln L(p) = \ln {}_nC_X + X \ln p + (n - X) \ln (1-p)$$

$$\frac{d}{dp} \ln L(p) = \frac{X}{p} - \frac{n-X}{1-p} = \frac{X - X\,p - n\,p + X\,p}{p\,(1-p)} = \frac{X - n\,p}{p\,(1-p)}$$

$$\frac{d}{dp} \ln L(\hat{p}) = 0 \qquad \Rightarrow \qquad \hat{p} = \frac{X}{n}.$$

2. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a Poisson distribution with mean λ, $\lambda > 0$. That is,

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, 3, \dots .$$

a) Obtain the method of moments estimator of λ, $\tilde{\lambda}$.

$$E(X) = \lambda \qquad \Rightarrow \qquad \tilde{\lambda} = \overline{X}$$

b) Obtain the maximum likelihood estimator of λ, $\hat{\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n} f(X_i; \lambda) = \prod_{i=1}^{n} \left(\frac{\lambda^{X_i} e^{-\lambda}}{X_i!} \right).$$

$$\ln L(\lambda) = \left(\sum_{i=1}^{n} X_i \right) \cdot \ln \lambda - n\lambda - \sum_{i=1}^{n} \ln(X_i!).$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{1}{\lambda} \cdot \left(\sum_{i=1}^{n} X_i \right) - n = 0. \qquad \Rightarrow \qquad \hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{n} = \overline{X}.$$

Let $\hat{\theta}$ be the maximum likelihood estimate (m.l.e.) of θ. Then the m.l.e. of any function $h(\theta)$ is $h(\hat{\theta})$. (The Invariance Principle)

c) Obtain the maximum likelihood estimator of $P(X = 2)$.

$$P(X = 2) = h(\lambda) = \frac{\lambda^2 e^{-\lambda}}{2!} \qquad \hat{\lambda} = \overline{X} \qquad h(\hat{\lambda}) = \frac{\overline{X}^2 e^{-\overline{X}}}{2!}.$$

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Geometric distribution with probability of "success" p, $0 < p < 1$. That is,

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \ldots .$$

a) Obtain the method of moments estimator of p, $\tilde{p}$.

$E(X) = 1/p$. $\quad \overline{X} = 1/\tilde{p}$ so $\tilde{p} = 1/\overline{X} = n \Big/ \sum_{i=1}^{n} X_i$.

b) Obtain the maximum likelihood estimator of p, $\hat{p}$.

$$L(p) = (1-p)^{\sum_{i=1}^{n} X_i - n}\; p^{\,n}$$

$$\ln L(p) = \left(\sum_{i=1}^{n} X_i - n\right) \ln(1-p) + n \ln p$$

$$\frac{d}{dp} \ln L(p) = \frac{n}{p} - \frac{\sum_{i=1}^{n} X_i - n}{1-p} = \frac{n - p\sum_{i=1}^{n} X_i}{p(1-p)}$$

$$\frac{d}{dp} \ln L(\hat{p}) = 0 \qquad \Rightarrow \qquad \hat{p} = n \Big/ \sum_{i=1}^{n} X_i = 1/\overline{X}.$$

$\hat{p} = \tilde{p}$ equals the number of successes, n, divided by the number of Bernoulli trials, $\sum_{i=1}^{n} X_i$;

c)* Is $\hat{p}$ an unbiased estimator for p?

Since $g(x) = 1/x$, $x \ge 1$, is strictly convex, and $\overline{X}$ is not a constant random variable, by Jensen's Inequality,

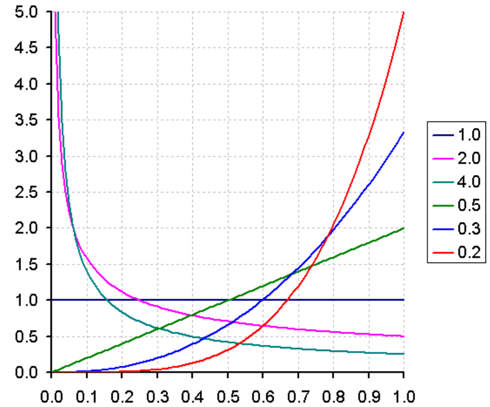
$$E(\hat{p}) = E(g(\overline{X})) > g(E(\overline{X})) = g(1/p) = p.$$

$\hat{p}$ is NOT an unbiased estimator for p.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} \cdot x^{\frac{1-\theta}{\theta}} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$0 < \theta < \infty$.



a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x;\theta)\, dx = \int_0^1 \left(x \cdot \frac{1}{\theta} \cdot x^{\frac{1-\theta}{\theta}} \right) dx.$$

$$= \int_0^1 \left(\frac{1}{\theta} \cdot x^{\frac{1}{\theta}} \right) dx = \frac{1}{\theta} \cdot \left(\frac{1}{\frac{1}{\theta} + 1} \cdot x^{\frac{1}{\theta} + 1} \right) \Bigg|_0^1 = \frac{1}{1+\theta}.$$

$$\overline{X} = \frac{1}{1+\theta}. \qquad \tilde{\theta} = \frac{1}{\overline{X}} - 1 = \frac{1-\overline{X}}{\overline{X}}.$$

b) Obtain the maximum likelihood estimator of θ, $\hat{\theta}$.

Likelihood function:

$$L(\theta) = \prod_{i=1}^{n} f_X(X_i;\theta) = \frac{1}{\theta^n} \cdot \left(\prod_{i=1}^{n} X_i \right)^{\frac{1-\theta}{\theta}}.$$

$$\ln L(\theta) = -n \cdot \ln\theta + \frac{1-\theta}{\theta} \cdot \sum_{i=1}^{n} \ln X_i = -n \cdot \ln\theta + \left(\frac{1}{\theta} - 1 \right) \cdot \sum_{i=1}^{n} \ln X_i.$$

$$\frac{d}{d\theta}\left(\ln L(\hat{\theta}) \right) = -\frac{n}{\hat{\theta}} - \frac{1}{\hat{\theta}^2} \cdot \sum_{i=1}^{n} \ln X_i = 0. \qquad \Rightarrow \qquad \hat{\theta} = -\frac{1}{n} \cdot \sum_{i=1}^{n} \ln X_i.$$

c) Suppose $n = 3$, and $x_1 = 0.2$, $x_2 = 0.3$, $x_3 = 0.5$. Compute the values of the method of moments estimate and the maximum likelihood estimate for θ.

$$\overline{X} = \frac{0.2 + 0.3 + 0.5}{3} = \frac{1}{3}. \qquad \tilde{\theta} = \frac{1 - \overline{X}}{\overline{X}} = \frac{1 - 1/3}{1/3} = \mathbf{2}.$$

$$\hat{\theta} = -\frac{1}{n} \cdot \sum_{i=1}^{n} \ln X_i = -\frac{1}{3} \cdot \left(\ln 0.2 + \ln 0.3 + \ln 0.5 \right) \approx \mathbf{1.16885}.$$

5. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $N(\theta_1, \theta_2)$, where $\Omega = \{ (\theta_1, \theta_2) : -\infty < \theta_1 < \infty, \ 0 < \theta_2 < \infty \}$. That is, here we let $\theta_1 = \mu$ and $\theta_2 = \sigma^2$.

a) Obtain the maximum likelihood estimator of θ_1, $\hat{\theta}_1$, and of θ_2, $\hat{\theta}_2$.

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} \exp\left[-\frac{(X_i - \theta_1)^2}{2\theta_2} \right]$$

$$= \left(\frac{1}{\sqrt{2\pi\theta_2}} \right)^n \exp\left[\frac{-\sum_{i=1}^{n} (X_i - \theta_1)^2}{2\theta_2} \right], \qquad (\theta_1, \theta_2) \in \Omega.$$

$$\ln L(\theta_1, \theta_2) = -\frac{n}{2} \ln(2\pi\theta_2) - \frac{\sum_{i=1}^{n} (X_i - \theta_1)^2}{2\theta_2}.$$

The partial derivatives with respect to θ_1 and θ_2 are

$$\frac{\partial(\ln L)}{\partial\theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(X_i - \theta_1)$$

and

$$\frac{\partial(\ln L)}{\partial\theta_2} = -\frac{n}{2\theta_2} + \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(X_i - \theta_1)^2 .$$

The equation $\partial(\ln L)/\partial\theta_1 = 0$ has the solution $\theta_1 = \overline{X}$.

Setting $\partial(\ln L)/\partial\theta_2 = 0$ and replacing θ_1 by $\overline{X}$ yields

$$\theta_2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 .$$

Therefore, the maximum likelihood estimators of $\mu = \theta_1$ and $\sigma^2 = \theta_2$ are

$$\hat{\theta}_1 = \overline{X} \quad \text{and} \quad \hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 .$$

b) Obtain the method of moments estimator of θ_1, $\tilde{\theta}_1$, and of θ_2, $\tilde{\theta}_2$.

$E[X] = \mu = \theta_1$.

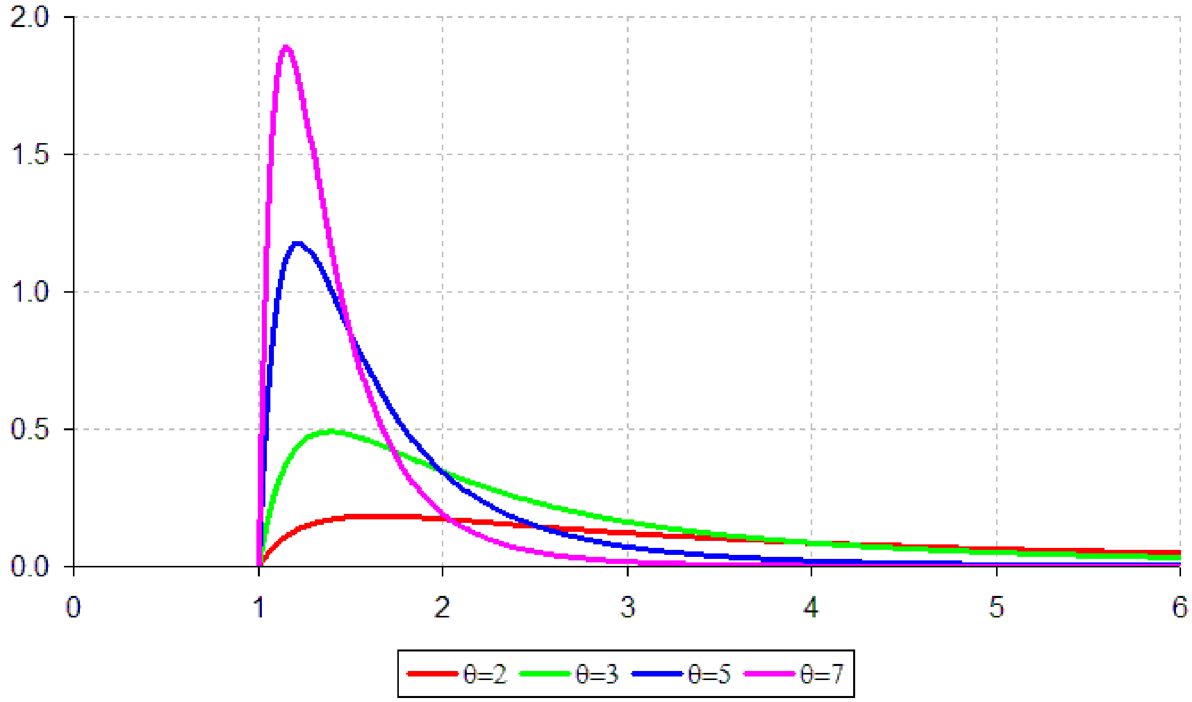
$E[X^2] = \operatorname{Var}[X] + E[X]^2 = \sigma^2 + \mu^2 = \theta_2 + \theta_1^2$.

Thus, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \tilde{\theta}_1$, $\qquad \overline{X^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \tilde{\theta}_2 + \tilde{\theta}_1^2$.

Therefore, $\tilde{\theta}_1 = \overline{X}$ and $\tilde{\theta}_2 = \overline{X^2} - \left(\overline{X}\right)^2$.

5½. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the distribution with probability density function

$$f_X(x) = f_X(x;\theta) = (\theta-1)^2 \cdot \frac{\ln x}{x^{\theta}}, \qquad x > 1, \qquad \theta > 1.$$



a) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

$$L(\theta) = \prod_{i=1}^{n} (\theta-1)^2 \cdot \frac{\ln x_i}{x_i^{\theta}}. \qquad \ln L(\theta) = 2n\ln(\theta-1) + \sum_{i=1}^{n} \ln\ln x_i - \theta \cdot \sum_{i=1}^{n} \ln x_i .$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{2n}{\theta-1} - \sum_{i=1}^{n} \ln x_i = 0. \qquad \Rightarrow \qquad \hat{\theta} = 1 + \frac{2n}{\sum_{i=1}^{n} \ln X_i}.$$

b) Suppose $n = 5$, and $x_1 = 5$, $x_2 = 1.2$, $x_3 = 2$, $x_4 = 12$, $x_5 = 1.5$.
Find the maximum likelihood estimate $\hat{\theta}$ of θ.

$$\sum_{i=1}^{5} \ln x_i = \ln 216 \approx 5.3753.$$

$$\hat{\theta} \approx 1 + \frac{10}{5.3753} \approx \mathbf{2.86}.$$

c) Suppose $\theta > 2$. Find the method of moments estimator $\tilde{\theta}$ of θ.

$$\mathrm{E(X)} = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{1}^{\infty} x \cdot (\theta - 1)^2 \cdot \frac{\ln x}{x^{\theta}}\, dx = \frac{(\theta-1)^2}{(\theta-2)^2}.$$

$$\frac{1}{n} \cdot \sum_{i=1}^{n} \mathrm{X}_i = \overline{\mathrm{X}} = \frac{(\theta-1)^2}{(\theta-2)^2}. \qquad \Rightarrow \qquad \tilde{\theta} = \frac{2\sqrt{\overline{\mathrm{X}}} - 1}{\sqrt{\overline{\mathrm{X}}} - 1}.$$

d) Suppose $n = 5$, and $x_1 = 5$, $x_2 = 1.2$, $x_3 = 2$, $x_4 = 12$, $x_5 = 1.5$.
Find the method of moments estimate $\tilde{\theta}$ of θ.

$$\bar{x} = 4.34.$$

$$\tilde{\theta} = \frac{2\sqrt{4.34} - 1}{\sqrt{4.34} - 1} \approx \mathbf{2.923}.$$

For fun:

e) Find $\mathrm{E}(\mathrm{X}^k)$, $k < \theta - 1$.

$$f_{\mathrm{X}}(x;\beta) = (\beta-1)^2 \cdot \frac{\ln x}{x^{\beta}}$$ is a probability density function, $\beta > 1$.

$$\Rightarrow \quad \int_1^{\infty} \frac{\ln x}{x^{\beta}}\,dx = \frac{1}{(\beta-1)^2}, \qquad \beta > 1.$$

$$\mathrm{E}(\mathrm{X}^k) = \int_{-\infty}^{\infty} x^k \cdot f_{\mathrm{X}}(x)\,dx = \int_1^{\infty} x^k \cdot (\theta-1)^2 \cdot \frac{\ln x}{x^{\theta}}\,dx$$

$$= (\theta-1)^2 \cdot \int_1^{\infty} \frac{\ln x}{x^{\theta-k}}\,dx = \frac{(\theta-1)^2}{(\theta-k-1)^2}.$$

For example, $$\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^1) = \frac{(\theta-1)^2}{(\theta-1-1)^2} = \frac{(\theta-1)^2}{(\theta-2)^2}.$$