STAT 400
UIUC

Examples for 6.4, Part 2

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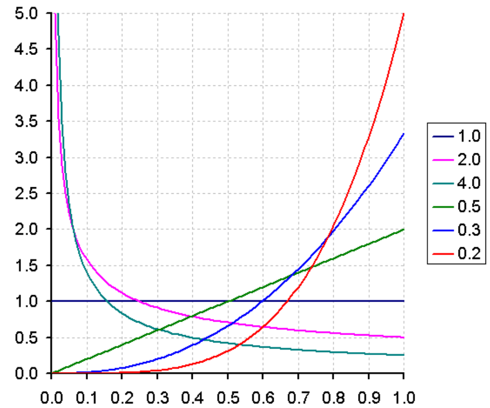
4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution with probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} \cdot x^{\frac{1-\theta}{\theta}} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$0 < \theta < \infty$.



Recall: Maximum likelihood estimator of θ is $\hat{\theta} = -\frac{1}{n} \cdot \sum_{i=1}^{n} \ln X_i$.

Method of moments estimator of θ is $\tilde{\theta} = \frac{1-\overline{X}}{\overline{X}} = \frac{1}{\overline{X}} - 1$. $E(X) = \frac{1}{1+\theta}$.

Def An estimator $\hat{\theta}$ is said to be **unbiased for θ** if $E(\hat{\theta}) = \theta$ for all θ.

d) Is $\hat{\theta}$ unbiased for θ? That is, does $E(\hat{\theta})$ equal θ?

Jensen's Inequality:

If g is convex on an open interval I and X is a random variable whose support is contained in I and has finite expectation, then

$$E[g(X)] \ge g[E(X)].$$

If g is strictly convex then the inequality is strict, unless X is a constant random variable.

e) Is $\tilde{\theta}$ unbiased for θ? That is, does $E(\tilde{\theta})$ equal θ?

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sample mean

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

sample variance

$$S^2 = \frac{1}{n-1} \sum \left(X_i - \overline{X} \right)^2$$

6. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with mean μ and variance σ^2. Show that the sample mean $\overline{X}$ and the sample variance S^2 are unbiased for μ and σ^2, respectively.

For an estimator $\hat{\theta}$ of θ, define the **Mean Squared Error** of $\hat{\theta}$ by

$$\text{MSE}(\hat{\theta}) = E[(\hat{\theta} - \theta)^2].$$

$$E[(\hat{\theta} - \theta)^2] = \left(E(\hat{\theta}) - \theta \right)^2 + \text{Var}(\hat{\theta}) = \left(\text{bias}(\hat{\theta}) \right)^2 + \text{Var}(\hat{\theta}).$$