4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from the distribution with probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} \cdot x^{\frac{1-\theta}{\theta}} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$0 < \theta < \infty$.

Recall: Maximum likelihood estimator of $\theta$ is $\hat{\theta} = -\frac{1}{n} \sum \ln X_i$.

Method of moments estimator of $\theta$ is $\tilde{\theta} = \frac{1}{\bar{X}} - 1$. $E(X) = \frac{1}{1+\theta}$.

**Def** An estimator $\hat{\theta}$ is said to be **unbiased for $\theta$** if $E(\hat{\theta}) = \theta$ for all $\theta$.
d) Is $\hat{\theta}$ unbiased for $\theta$? That is, does $E(\hat{\theta}) = \theta$?

\[
E(\ln X_1) = \int_{-\infty}^{\infty} \ln x \cdot f_X(x; \theta) \, dx = \int_{0}^{1} \left( \ln x \cdot \frac{1-\theta}{\theta} \right) \, dx.
\]

Integration by parts:

\[
\int_{a}^{b} u \, dv = \left[ u \cdot v \right]_{a}^{b} - \int_{a}^{b} v \, du
\]

Choice of $u$:

$U \rightarrow \text{Logarithmic}$

$A \rightarrow \text{Algebraic}$

$T \rightarrow \text{Trigonometric}$

$E \rightarrow \text{Exponential}$

\[u = \ln x, \quad dv = \frac{1-\theta}{\theta} \cdot x \, dx = \frac{1}{\theta} \cdot x \, dx,\]

\[du = \frac{1}{x} \, dx, \quad v = x^{1-\theta}.
\]

\[
E(\ln X_1) = \int_{0}^{1} \left( \ln x \cdot \frac{1}{\theta} \right) \, dx = \left[ \ln x \cdot x^{1/\theta} \right]_{0}^{1} - \int_{0}^{1} \left( \frac{1}{x} \cdot x^{1/\theta} \right) \, dx
\]

\[
= -\int_{0}^{1} \left( \frac{1}{x} \cdot x^{1/\theta} \right) \, dx = -\int_{0}^{1} x^{1-1/\theta} \, dx = -\left[ \frac{x^{1/\theta}}{1/\theta} \right]_{0}^{1} = -\theta.
\]

Therefore,

\[
E(\hat{\theta}) = -\frac{1}{n} \cdot \sum_{i=1}^{n} E(\ln X_i) = -\frac{1}{n} \cdot \sum_{i=1}^{n} (-\theta) = \theta,
\]

that is, $\hat{\theta}$ is an unbiased estimator for $\theta$. 
Jensen’s Inequality:

If \( g \) is convex on an open interval \( I \) and \( X \) is a random variable whose support is contained in \( I \) and has finite expectation, then

\[
E[g(X)] \geq g(E(X)).
\]

If \( g \) is strictly convex then the inequality is strict, unless \( X \) is a constant random variable.

\[
\begin{align*}
\Rightarrow & \quad E(X^2) \geq [E(X)]^2 & \iff \text{Var}(X) \geq 0 \\
\Rightarrow & \quad E(e^{tX}) \geq e^{tE(X)} & \Rightarrow M_X(t) \geq e^{t\mu} \\
\Rightarrow & \quad E\left(\frac{1}{X}\right) \geq \frac{1}{E(X)} & \text{for a positive random variable } X \\
\Rightarrow & \quad E[X^3] \geq [E(X)]^3 & \text{for a non-negative random variable } X \\
\Rightarrow & \quad E[\ln X] \leq \ln E(X) & \text{for a positive random variable } X \\
\Rightarrow & \quad E\left(\sqrt{X}\right) \leq \sqrt{E(X)} & \text{for a non-negative random variable } X \\
\end{align*}
\]

e) Is \( \tilde{\theta} \) unbiased for \( \theta \)? That is, does \( E(\tilde{\theta}) \) equal \( \theta \)?

Since \( g(x) = \frac{1-x}{x} = \frac{1}{x} - 1, \quad 0 < x < 1, \) is strictly convex, and \( \bar{X} \) is not a constant random variable, by Jensen’s Inequality,

\[
E(\tilde{\theta}) = E(g(\bar{X})) > g(E(\bar{X})) = \theta.
\]

\( \tilde{\theta} \) is NOT an unbiased estimator for \( \theta \).
6. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a population with mean $\mu$ and variance $\sigma^2$. Show that the sample mean $\bar{X}$ and the sample variance $S^2$ are unbiased for $\mu$ and $\sigma^2$, respectively.

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

$$E(X_1 + X_2 + \ldots + X_n) = n \cdot \mu \quad \Rightarrow \quad E(\bar{X}) = \mu \quad \checkmark$$

$$E(X^2) = \text{Var}(X) + [E(X)]^2 = \mu^2 + \sigma^2.$$  

$$\text{Var}(X_1 + X_2 + \ldots + X_n) = n \cdot \sigma^2 \quad \Rightarrow \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

$$E\left(\left(\bar{X}\right)^2\right) = \text{Var}(\bar{X}) + [E(\bar{X})]^2 = \mu^2 + \frac{\sigma^2}{n}.$$  

$$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2 = \frac{1}{n-1} \left[ \sum X_i^2 - n \cdot (\bar{X})^2 \right]$$

$$E(S^2) = \frac{1}{n-1} \left[ \sum E(X_i^2) - n \cdot E(\left(\bar{X}\right)^2) \right]$$

$$= \frac{1}{n-1} \left[ n \cdot \left( \mu^2 + \sigma^2 \right) - n \cdot \left( \mu^2 + \frac{\sigma^2}{n} \right) \right] = \sigma^2 \quad \checkmark$$

For an estimator $\hat{\theta}$ of $\theta$, define the **Mean Squared Error** of $\hat{\theta}$ by

$$\text{MSE}(\hat{\theta}) = E \left[ (\hat{\theta} - \theta)^2 \right].$$

$$E \left[ (\hat{\theta} - \theta)^2 \right] = (E(\hat{\theta}) - \theta)^2 + \text{Var}(\hat{\theta}) = (\text{bias}(\hat{\theta}))^2 + \text{Var}(\hat{\theta}).$$
7. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a distribution with probability density function

$$f_X(x) = f_X(x; \theta) = \frac{1+\theta x}{2}, \quad -1 < x < 1, \quad -1 < \theta < 1.$$ 

a) Obtain the method of moments estimator of $\theta$, $\tilde{\theta}$.

$$\mu = E(X) = \int_{-1}^{1} x \cdot \frac{1+\theta x}{2} \, dx = \left( \frac{x^2}{4} + \frac{\theta x^3}{6} \right) \bigg|_{-1}^{1} = \frac{\theta}{3}.$$ 

$$\overline{X} = \frac{\theta}{3} \quad \Rightarrow \quad \tilde{\theta} = 3 \overline{X}.$$ 

b) Is $\tilde{\theta}$ an unbiased estimator for $\theta$? Justify your answer.

$$E(\tilde{\theta}) = E(3 \overline{X}) = 3 E(\overline{X}) = 3 \mu = 3 \frac{\theta}{3} = \theta.$$ 

$$\Rightarrow \quad \tilde{\theta} \text{ an unbiased estimator for } \theta.$$ 

c) Find $\text{Var}(\tilde{\theta})$.

$$E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{1+\theta x}{2} \, dx = \left( \frac{x^3}{6} + \frac{\theta x^4}{8} \right) \bigg|_{-1}^{1} = \frac{1}{3}.$$ 

$$\sigma^2 = \text{Var}(X) = \frac{1}{3} - \left( \frac{\theta}{3} \right)^2 = \frac{3-\theta^2}{9}.$$ 

$$\text{Var}(\tilde{\theta}) = 9 \text{Var}(\overline{X}) = 9 \cdot \frac{\sigma^2}{n} = \frac{3-\theta^2}{n}.$$ 

$$\Rightarrow \quad \text{MSE}(\tilde{\theta}) = \frac{3-\theta^2}{n}.$$
8. Let \( X_1, X_2 \) be a random sample of size \( n = 2 \) from a distribution with probability density function

\[
f_X(x) = f_X(x; \theta) = \frac{1 + \theta x}{2}, \quad -1 < x < 1, \quad -1 < \theta < 1.
\]

Find the maximum likelihood estimator \( \hat{\theta} \) of \( \theta \).

\[
L(\theta) = \frac{1 + \theta x_1}{2} \cdot \frac{1 + \theta x_2}{2} = \frac{1 + \theta (x_1 + x_2) + \theta^2 x_1 x_2}{4}
\]

\( L(\theta) \) is a parabola with vertex at \( \frac{-b}{2a} = \frac{-(x_1 + x_2)}{2x_1 x_2} \).

**Case 1:** \( a = x_1 x_2 > 0 \). Parabola has its “antlers” up.

\[ \Rightarrow \quad \text{The vertex is the minimum.} \]

**Subcase 1:** \( x_1 > 0, \ x_2 > 0 \). Vertex = \( \frac{-x_1 + x_2}{2x_1 x_2} < 0 \).

Maximum of \( L(\theta) \) on \(-1 < \theta < 1\) is at \( \hat{\theta} = 1 \).

**Subcase 2:** \( x_1 < 0, \ x_2 < 0 \). Vertex = \( \frac{-x_1 + x_2}{2x_1 x_2} > 0 \).

Maximum of \( L(\theta) \) on \(-1 < \theta < 1\) is at \( \hat{\theta} = -1 \).

**Case 2:** \( a = x_1 x_2 < 0 \). Parabola has its “antlers” down.

\[ \Rightarrow \quad \text{The vertex is the maximum.} \]

Vertex is at \( \frac{x_1 + x_2}{2x_1 x_2} \).
Subcase 1: \( -\frac{x_1 + x_2}{2x_1x_2} > 1 \). That is, \( x_2 > -\frac{x_1}{2x_1+1} \).

Maximum of \( L(\theta) \) on \( -1 < \theta < 1 \) is at \( \hat{\theta} = 1 \).

Subcase 2: \( -\frac{x_1 + x_2}{2x_1x_2} < -1 \). That is, \( x_2 < \frac{x_1}{2x_1-1} \).

Maximum of \( L(\theta) \) on \( -1 < \theta < 1 \) is at \( \hat{\theta} = -1 \).

Subcase 3: \( -1 < -\frac{x_1 + x_2}{2x_1x_2} < 1 \).

Maximum of \( L(\theta) \) on \( -1 < \theta < 1 \) is at \( \hat{\theta} = -\frac{X_1 + X_2}{2X_1X_2} \).

Pink \( \hat{\theta} = 1 \).

Purple \( \hat{\theta} = -1 \).

Green \( \hat{\theta} = -\frac{X_1 + X_2}{2X_1X_2} \).
Let \( X_1, X_2, \ldots, X_n \) be a random sample from the distribution with probability density function
\[
f(x) = 4 \theta x^3 e^{-\theta x^4} \quad x > 0 \quad \theta > 0.
\]

a) Obtain the maximum likelihood estimator of \( \theta, \hat{\theta} \).

\[
L(\theta) = \prod_{i=1}^{n} 4 \theta x_i^3 e^{-\theta x_i^4}
\]

\[
\ln L(\theta) = n \cdot \ln \theta + \sum_{i=1}^{n} \ln \left(4 x_i^3 \right) - \theta \cdot \sum_{i=1}^{n} x_i^4
\]

\[
\left( \ln L(\theta) \right)' = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^4 = 0 \quad \Rightarrow \quad \hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i^4}.
\]

b) Find \( E(X^k) \), \( k > -4 \).

\[
E(X^k) = \int_{0}^{\infty} x^k 4 \theta x^3 e^{-\theta x^4} dx = \int_{0}^{\infty} \left( \frac{u}{\theta} \right)^{k/4} e^{-u} du = \frac{1}{\theta^{k/4}} \Gamma \left( \frac{k}{4} + 1 \right).
\]

c) Find the method of moments estimator of \( \theta, \tilde{\theta} \).

\[
E(X) = E(X^1) = \frac{1}{\theta^{1/4}} \Gamma \left( \frac{1}{4} + 1 \right) = \frac{1}{\theta^{1/4}} \Gamma(1.25) \approx \frac{0.9064}{\theta^{1/4}}.
\]

\[
\bar{X} = \frac{\Gamma(1.25)}{\theta^{1/4}}.
\]

\[
\tilde{\theta} = \left( \frac{\Gamma(1.25)}{\bar{X}} \right)^4 \approx \frac{0.675}{(\bar{X})^4}.
\]