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Answers for 5.5, 7.1

¼. Let Z be a $N(0, 1)$ standard normal random variable.

Then $X = Z^2$ has a chi-square distribution with 1 degree of freedom.

$$M_X(t) = E(e^{tZ^2}) = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(z^2/2\right)\cdot\left(1-2t\right)} dz = \frac{1}{(1-2t)^{1/2}}, \qquad t < 1/2,$$

since $\frac{(1-2t)^{1/2}}{\sqrt{2\pi}} e^{-\left(z^2/2\right)\cdot\left(1-2t\right)}$ is the p.d.f. of a $N(0, \frac{1}{1-2t})$ random variable.

$\Rightarrow$ X has a $\chi^2(1)$ distribution.

½. Let X and Y be be two independent χ^2 random variables with m and n degrees of freedom, respectively. Then $W = X + Y$ has a chi-square distribution with $m + n$ degrees of freedom.

$$M_X(t) = \frac{1}{(1-2t)^{m/2}}, \quad t < 1/2, \qquad M_Y(t) = \frac{1}{(1-2t)^{n/2}}, \quad t < 1/2.$$

$$M_W(t) = M_X(t)\cdot M_Y(t) = \frac{1}{(1-2t)^{(m+n)/2}}, \quad t < 1/2.$$

$\Rightarrow$ W has a $\chi^2(m+n)$ distribution.

1. A manufacturer of TV sets wants to find the average selling price of a particular model. A random sample of 25 different stores gives the mean selling price as \$342 with a sample standard deviation of \$14. Assume the prices are normally distributed. Construct a 95% confidence interval for the mean selling price of the TV model.

σ is unknown. $n = 25$ – small. The confidence interval : $\overline{X} \pm t_{\alpha/2} \cdot \dfrac{s}{\sqrt{n}}$.

$n - 1 = 25 - 1 = 24$ degrees of freedom.

95% confidence level, $\alpha = 0.05$, $\alpha/2 = 0.025$, $t_{\alpha/2}(24) = 2.064$.

$342 \pm 2.064 \dfrac{14}{\sqrt{25}}$ **342 ± 5.78** **(336.22 ; 347.78)**

2. The following random sample was obtained from $N(\mu, \sigma^2)$ distribution:

16 12 18 13 21 15 8 17

a) Compute the sample mean and the sample standard deviation.

$$\bar{x} = \frac{\sum x}{n} = \frac{16+12+18+13+21+15+8+17}{8} = \frac{120}{8} = 15.$$

x	x^2
16	256
12	144
18	324
13	169
21	441
15	225
8	64
17	289
	1912

OR

x	$x-\bar{x}$	$(x-\bar{x})^2$
16	1	1
12	-3	9
18	3	9
13	-2	4
21	6	36
15	0	0
8	-7	49
17	2	4
	0	112

$$s^2 = \frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1} = \frac{1912 - \frac{(120)^2}{8}}{7} = 16. \qquad s^2 = \frac{\sum (x-\bar{x})^2}{n-1} = \frac{112}{7} = 16.$$

$$s = \sqrt{s^2} = \sqrt{16} = 4.$$

b) Construct a 95% confidence interval for μ.

Confidence interval: $\bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$. $n - 1 = 7$ degrees of freedom.

95% confidence level, $\alpha = 0.05$, $\alpha/2 = 0.025$, $t_{0.025} = 2.365$.

$15 \pm 2.365 \cdot \frac{4}{\sqrt{8}}$ **15 ± 3.3446** **$(11.6554, 18.3446)$**

b½) Construct a 90% confidence interval for μ.

Confidence interval: $\bar{x} \pm t_{\alpha/2} \cdot \dfrac{s}{\sqrt{n}}$. $n - 1 = 7$ degrees of freedom.

90% confidence level, $\alpha = 0.10$, $\alpha/2 = 0.05$, $t_{0.05} = 1.895$.

$15 \pm 1.895 \cdot \dfrac{4}{\sqrt{8}}$ **15 ± 2.68** **$(12.32, 17.68)$**

c) Construct a 90% confidence upper bound for μ.

$\left(-\infty,\ \bar{X} + t_{\alpha} \dfrac{s}{\sqrt{n}}\right)$ $n - 1 = 7$ degrees of freedom $t_{0.10} = 1.415$.

$\left(-\infty,\ 15 + 1.415 \cdot \dfrac{4}{\sqrt{8}}\right)$ **$(-\infty\ ;\ 17)$**

d) Construct a 99% confidence lower bound for μ.

$\left(\bar{X} - t_{\alpha} \dfrac{s}{\sqrt{n}},\ \infty\right)$ $n - 1 = 7$ degrees of freedom $t_{0.01} = 2.998$.

$\left(15 - 2.998 \cdot \dfrac{4}{\sqrt{8}},\ \infty\right)$ **$(10.76\ ;\ \infty)$**