STAT 400 UIUC | **Answers for 9.1** | Stepanov Dalpiaz

Pearson's χ^2 Test for Goodness of Fit (Based on Large n)

A random sample of size n is classified into k categories or cells.

Let $Y_1, Y_2, \dots, Y_k$ denote the respective cell frequencies. $\sum_{i=1}^{k} Y_i = n$.

Denote the cell probabilities by $p_1, p_2, \dots, p_k$.

$$H_0: p_1 = p_{10},\ p_2 = p_{20},\ \dots,\ p_k = p_{k0}. \qquad \sum_{i=1}^{k} p_{i0} = 1.$$

	1	2	…	k	Total
Observed frequency O	Y_1	Y_2	…	Y_k	n
Probability under H_0	p_{10}	p_{20}	…	p_{k0}	1
Expected frequency E under H_0	$n\,p_{10}$	$n\,p_{20}$	…	$n\,p_{k0}$	n

Test Statistic:

$$Q_{k-1} = \sum_{i=1}^{k} \frac{(Y_i - n p_{i0})^2}{n p_{i0}} = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \sum_{cells} \frac{(O - E)^2}{E}$$

Rejection Region:

Reject H_0 if $Q_{k-1} \ge \chi^2_{\alpha}$,

d.f. = k – 1 = (number of cells) – 1

Pearson's χ^2 test is an approximate test that is valid only for large samples.

As a rule of thumb, n should be large enough so that expected frequency of each cell is at least 5.

1. Alex buys a package of Sour Jelly Beans. On the package, it says that 50% of all jelly beans are lemon, 30% are cherry, and 20% are lime. When Alex opens the package, he finds 15 lemon, 9 cherry and 12 lime jelly beans. Is there enough evidence to conclude that the true proportions are different from the ones listed on the package? Use $\alpha = 0.05$.

a) State H_0 and H_1.

$H_0: p_1 = 0.50, \; p_2 = 0.30, \; p_3 = 0.20.$ $\qquad H_1:$ not H_0.

b) Find the expected frequencies under H_0.

$n = 15 + 9 + 12 = 36.$

$E_1 = n\,p_{10} = 36 \times 0.50 = \mathbf{18}.$

$E_2 = n\,p_{20} = 36 \times 0.30 = \mathbf{10.8}.$

$E_3 = n\,p_{30} = 36 \times 0.20 = \mathbf{7.2}.$

c) Calculate the values of the Q test statistic.

	1	2	3	Total
O	15	9	12	36
E	18	10.8	7.2	36
$\frac{(O-E)^2}{E}$	$\frac{(15-18)^2}{18}$	$\frac{(9-10.8)^2}{10.8}$	$\frac{(12-7.2)^2}{7.2}$	
	0.5	0.3	3.2	**4.0**

d) Find the critical value χ^2_α.

$3 - 1 = 2$ degrees of freedom. $\qquad \chi^2_{0.05}(2) = \mathbf{5.991}.$

e) State your decision (Accept H_0 or Reject H_0) at $\alpha = 0.05$.

$Q_2 = 4.0 < 5.991 = \chi^2_\alpha.$ $\qquad$ **Do Not Reject H_0.**

2. The financial manager in charge of accounts receivable department is concerned about the current economic slowdown, because customers sometimes wait longer to pay their bills. She wishes to check on this year's performance of the department by comparing the current outstanding accounts with records from the past few years. Historical records show the following percentages in the respective classifications:

Age of Accounts Receivable	**Percent of Total Receivables**
Less than 30 days	50%
Between 30 and 60 days	25%
Between 60 and 90 days	15%
Over 90 days	10%

To avoid the time required for a complete audit of the accounts receivable, the financial manager chooses a random sample of 60 accounts and finds 24, 18, 15, and 3 accounts, respectively, in the above categories.

$H_0: \ p_1 = 0.50, \ \ p_2 = 0.25, \ \ p_3 = 0.15, \ \ p_4 = 0.10.$

Perform the appropriate test at the $\alpha = 0.05$ level of significance.

O:	24	18	15	3
E:	$60 \cdot 0.50 = 30$	$60 \cdot 0.25 = 15$	$60 \cdot 0.15 = 9$	$60 \cdot 0.10 = 6$
$\frac{(O-E)^2}{E}$:	$\frac{(24-30)^2}{30}$	$\frac{(18-15)^2}{15}$	$\frac{(15-9)^2}{9}$	$\frac{(3-6)^2}{6}$
	1.2	0.6	4.0	1.5

$$Q = \sum_{cells} \frac{(O-E)^2}{E} = 1.2 + 0.6 + 4.0 + 1.5 = \mathbf{7.3}.$$

$k - 1 = 4 - 1 = 3$ d.f. $\qquad \chi^2_{0.05} = \mathbf{7.815}.$

Rejection Region: "Reject H_0 if $Q \ge \chi^2_{\alpha}$"

$7.3 = Q < \chi^2_{\alpha} = 7.815.$ $\qquad$ **Do Not Reject H_0 at $\alpha = 0.05$.**

($\chi^2_{0.10} = 6.25.$ $\qquad$ Reject H_0 at $\alpha = 0.10$.) $\qquad$ (P-value = 0.062926)

3. The developers of a math proficiency exam to be used at Anytown State University believe that 60% of all incoming freshmen will be able to pass the exam. In a random sample of 200 incoming freshmen, 105 pass the exam. Does this contradict the claim of the developers?

a) Use $\alpha = 0.05$ to perform a goodness-of-fit test

$$H_0: \; p_1 = 0.60, \; p_2 = 0.40 \quad \text{vs.} \quad H_1: \; H_0 \text{ is not true.}$$

$E_1 = n\,p_{10} = 200 \times 0.60 = \mathbf{120}$. $\qquad$ $E_2 = n\,p_{20} = 200 \times 0.40 = \mathbf{80}$.

$$Q_1 = \frac{(105-120)^2}{120} + \frac{(95-80)^2}{80} = 1.875 + 2.8125 = \mathbf{4.6875}.$$

$2 - 1 = 1$ degree of freedom. $\qquad$ $\chi^2_{0.05}(1) = \mathbf{3.841}$.

$Q_1 = 4.6875 > 3.841 = \chi^2_\alpha$. $\qquad$ **Reject H_0**.

b) Use $\alpha = 0.05$ to perform a large-sample test

$$H_0: \; p = 0.60 \quad \text{vs.} \quad H_1: \; p \neq 0.60.$$

$\hat{p} = \frac{105}{200} = 0.525$. $\qquad$ $Z = \dfrac{0.525 - 0.60}{\sqrt{\dfrac{0.60 \cdot 0.40}{200}}} = \mathbf{-2.165}$.

$\pm z_{0.025} = \mathbf{\pm 1.960}$.

$Z = -2.165 < -1.960 = -z_{\alpha/2}$. $\qquad$ **Reject H_0**.

c) Compare the test statistics Q_1 (part (a)) and Z (part (b)). Compare the critical values $\chi^2_{0.05}$ (part (a)) and $\pm z_{0.025}$ (part (b)).

$$Q_1 \text{ (part (a))} = [\, Z \text{ (part (b))} \,]^2$$

$$\chi^2_{0.05} \text{ (part (a))} = [\, \pm z_{0.025} \text{ (part (b))} \,]^2$$