STAT 400
Exam I

Continuous RVs

Fall 2017
Practice

1. Suppose a random variable X has the following probability density function:

$$f(x) = \sin x, \qquad 0 < x < \frac{\pi}{2}, \qquad \text{zero otherwise.}$$

a) Find $P(X < \frac{\pi}{4})$.

b) Find $\mu = E(X)$.

c) Find the median of the probability distribution of X.

2. Suppose a random variable X has the following probability density function:

$$f(x) = x e^{x}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find $P(X < \frac{1}{2})$.

b) Find $\mu = E(X)$.

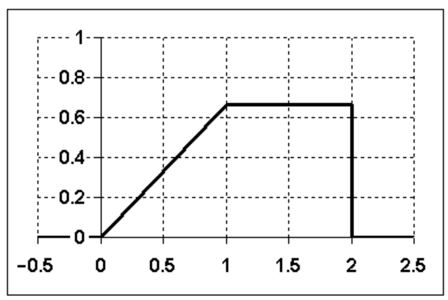
c) Find the moment-generating function of X, $M_X(t)$.

3. Let X have the probability density function:

$f(x) = \frac{2}{3} \cdot x$ for $0 \le x \le 1$,

$f(x) = \frac{2}{3}$ for $1 < x \le 2$,

$f(x) = 0$ otherwise.



a) Verify that $f(x)$ is a probability density function.

b) Find $P(X \le 0.5)$.

c) Find $P(X \le 1.5)$.

d) Find $P(X = 1.25)$.

e) Find the cumulative distribution function $F(x) = P(X \le x)$.

f) Find the median of the probability distribution of X.

g) Compute the expected value of X, $E(X) = \mu_X$.

4. The heights of adult males in Neverland are normally distributed with mean of 69 inches and standard deviation of 5 inches.

a) What proportion of adult males in Neverland are taller than 6 feet (72 inches)?

b) What proportion of adult males are between 5 and 6 feet tall.

c) How tall must a male be to be among the tallest 10% of the population?

d) How "tall" must a male be to be among the shortest third of the population?

5. The lifetime of a certain type of television tube is normally distributed with mean 3.8 years and standard deviation of 1.2 years.

a) Suppose that the tube is guaranteed for two years. What proportion of TVs will require a new tube before the guarantee expires?

b) If the company wishes to set the warranty period so that only 4% of the tubes would need replacement while under warranty, how long a warranty must be set?

c) What proportion of TVs will last for over 5 years?

6. The lifetimes of a certain type of light bulbs follow a normal distribution. If 90% of the bulbs have lives exceeding 2000 hours and 3% have lives exceeding 6000 hours, what are the mean and standard deviation of the lifetimes of this particular type of light bulbs.

7. Suppose the average daily temperature [in degrees Fahrenheit] in July in Anytown is a random variable T with mean $\mu_{\mathrm{T}} = 86$ and standard deviation $\sigma_{\mathrm{T}} = 9$. What is the probability that the daily temperature in July in Anytown is above 33 degrees Celsius? (Assume that T is a normal random variable.)

Celsius $\rightarrow$ Fahrenheit $F = \frac{9}{5} \cdot C + 32$ Fahrenheit $\rightarrow$ Celsius $C = \frac{5}{9} \cdot (F - 32)$

8. If the moment-generating function of X is $M(t) = \exp(166t + 200t^2)$, find

a) The mean of X.

b) The variance of X.

c) $P(170 < X < 200)$.

d) $P(148 < X < 172)$.

From the textbook:

9.	**3.1-8 (a)**	**3.3-2 (a), 3.3-4 (a)**	**3.2-2 (a)**
10.	**3.1-8 (b)**	**3.3-2 (b), 3.3-4 (b)**	**3.2-2 (b)**
11.	**3.1-8 (c)**	**3.3-2 (c), 3.3-4 (c)**	**3.2-2 (c)**
12.	**3.1-10**	**3.3-8**	**3.2-8**
13.	**3.1-4**	**3.4-4**	**3.3-4**
14.	**3.2-2**	**3.4-8**	**3.3-8**
15.	**3.3-6**	**3.6-6**	**5.2-6**
16.	**3.3-11**	**3.6-14**	**5.2-14**

Answers:

1. Suppose a random variable X has the following probability density function:

$$f(x) = \sin x, \qquad 0 < x < \frac{\pi}{2}, \qquad \text{zero otherwise.}$$

a) Find $P(X < \frac{\pi}{4})$.

$$P(X < \frac{\pi}{4}) = \int_0^{\pi/4} \sin x \, dx = \left(-\cos x\right)\Big|_0^{\pi/4} = 1 - \cos\frac{\pi}{4} = 1 - \frac{\sqrt{2}}{2} \approx 0.292893.$$

b) Find $\mu = E(X)$.

$$\mu = E(X) = \int_0^{\pi/2} x \cdot \sin x \, dx = \left(-x \cdot \cos x + \sin x\right)\Big|_0^{\pi/2} = \mathbf{1}.$$

c) Find the median of the probability distribution of X.

$$F(x) = P(X \le x) = \int_0^{x} \sin y \, dy = 1 - \cos x, \qquad 0 < x < \frac{\pi}{2}.$$

Median: $F(m) = P(X \le m) = \frac{1}{2}$.

$$\Rightarrow \quad \cos m = \frac{1}{2}. \qquad m = \boldsymbol{\frac{\pi}{3}}.$$

2. Suppose a random variable X has the following probability density function:

$$f(x) = x e^{x}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find $\mathrm{P}(\mathrm{X} < \frac{1}{2})$.

$$\mathrm{P}(\mathrm{X} < \tfrac{1}{2}) = \int_0^{1/2} x e^{x}\,dx = \left[x e^{x} - e^{x} \right] \Big|_0^{1/2} = 1 - \frac{\sqrt{e}}{2} \approx 0.175639.$$

b) Find $\mu = \mathrm{E}(\mathrm{X})$.

$$\mu = \mathrm{E}(\mathrm{X}) = \int_0^{1} x \cdot x e^{x}\,dx = \left[x^2 e^{x} - 2 x e^{x} + 2 e^{x} \right] \Big|_0^{1} = \boldsymbol{e - 2}.$$

c) Find the moment-generating function of X, $\mathrm{M}_{\mathrm{X}}(t)$.

$$\mathrm{M}_{\mathrm{X}}(t) = \int_0^{1} e^{tx} \cdot x e^{x}\,dx = \int_0^{1} x e^{(t+1)x}\,dx$$

$$= \left[\frac{1}{t+1} x e^{(t+1)x} - \frac{1}{(t+1)^2} e^{(t+1)x} \right] \Bigg|_0^{1}$$

$$= \frac{1}{t+1} e^{t+1} - \frac{1}{(t+1)^2} e^{t+1} + \frac{1}{(t+1)^2}$$

$$= \frac{t e^{t+1} + 1}{(t+1)^2}, \qquad t \neq -1.$$

$$\mathrm{M}_{\mathrm{X}}(-1) = \int_0^{1} x\,dx = \frac{1}{2}.$$

3. a) $f(x) \geq 0$, total area under the graph of $f(x)$ is 1.

b) $P(X \leq 0.5) = \mathbf{1/12}$.

c) $P(X \leq 1.5) = \mathbf{2/3}$.

d) $P(X = 1.25) = \mathbf{0}$.

e) $F(x) = \mathbf{0}$ for $x \leq 0$,

$F(x) = \frac{1}{3} \cdot x^2$ for $0 \leq x \leq 1$,

$F(x) = \frac{2}{3} \cdot x - \frac{1}{3}$ for $1 \leq x \leq 2$,

$F(x) = \mathbf{1}$ for $x \geq 2$.

f) median = **1.25**.

g) $E(X) = \mathbf{11/9}$.

4. The heights of adult males in Neverland are normally distributed with mean of 69 inches and standard deviation of 5 inches.

a) What proportion of adult males in Neverland are taller than 6 feet (72 inches)?

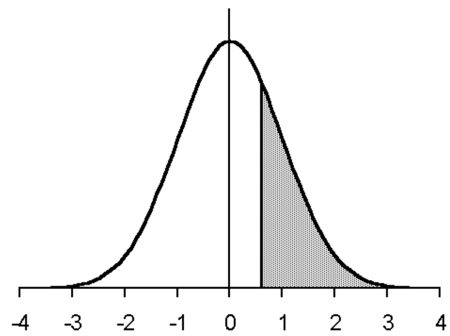
$$P(X > 72) = P\left(Z > \frac{72-69}{5}\right)$$

$= P(Z > 0.60)$

= area to the right of 0.60

$= 1 - \Phi(0.60)$

$= 1 - 0.7257 =$ **0.2743**.



b) What proportion of adult males are between 5 and 6 feet tall.

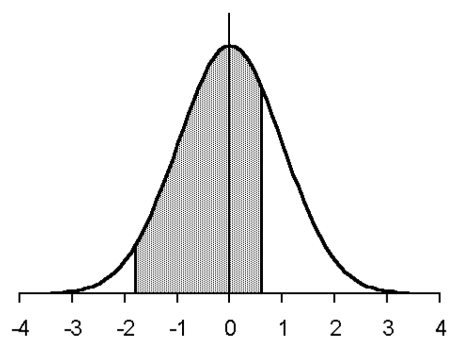
$$P(60 < X < 72) = P\left(\frac{60-69}{5} < Z < \frac{72-69}{5}\right)$$

$= P(-1.80 < Z < 0.60)$

= area between -1.80 and 0.60

$= \Phi(0.60) - \Phi(-1.80)$

$= 0.7257 - 0.0359 =$ **0.6898**.



c) How tall must a male be to be among the tallest 10% of the population?

Need x such that $P(X > x) = 0.10$.

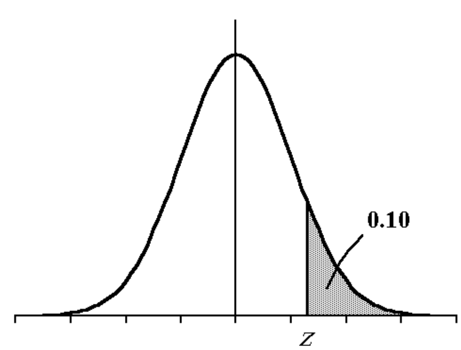
Need z such that $P(Z > z) = 0.10$.

$\Phi(z) = 1 - 0.10 = 0.90$.

$z =$ **1.28**.

$$\frac{x-\mu}{\sigma} = z \qquad \frac{x-69}{5} = 1.28.$$

$x = 69 + 5 \cdot 1.28 =$ **75.4** inches.



A male must to be **over 75.4 inches** tall.

d) How "tall" must a male be to be among the shortest third of the population?

Need x such that $P(X < x) = 0.3333$.

Need z such that $P(Z < z) = 0.3333$.

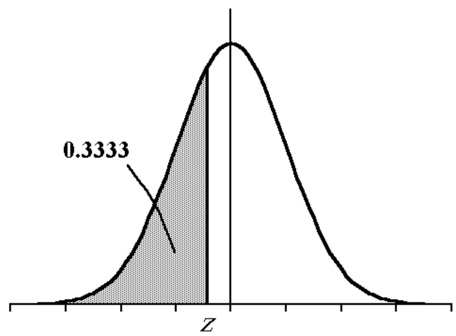
$\Phi(z) = 0.3333$.

$z = \mathbf{-0.43}$.

$\dfrac{x-\mu}{\sigma} = z \qquad \dfrac{x-69}{5} = -0.43$.

$x = 69 + 5 \cdot (-0.43) = \mathbf{66.85}$ inches.

A male must to be **under 66.85 inches** "tall."



5. $\mu = 3.8$ years, $\sigma = 1.2$ years.

a) $P(X < 2) = P(Z < -1.50) = 1 - 0.9332 = \mathbf{0.0668}$.

b) $z = -1.75$; $x = 3.8 + 1.2\,(-1.75) = \mathbf{1.7}$ years.

c) $P(X > 5) = P(Z > 1.00) = 1 - 0.8413 = \mathbf{0.1587}$.

6. $P(X > 2000) = 0.90$ $\qquad P(Z > -1.28) = 0.90$

$P(X > 6000) = 0.03$ $\qquad P(Z > 1.88) = 0.03$

$2000 = \mu - 1.28\,\sigma$

$6000 = \mu + 1.88\,\sigma$

$\mu \approx 3620.253$ hours, $\qquad \sigma \approx 1265.823$ hours.

7. $33°\text{C} = \frac{9}{5} \cdot 33 + 32 = 91.4$ degrees Fahrenheit.

$$P(T > 91.4) = P\left(Z > \frac{91.4 - 86}{9}\right) = P(Z > 0.60) = 1 - 0.7257 = \mathbf{0.2743}.$$

OR

Let X = daily temperature [in degrees Celsius] in July in Anytown.

Then $X = \frac{5}{9} \cdot (T - 32)$, X has Normal distribution,

$$\mu_X = \frac{5}{9} \cdot (\mu_T - 32) = \frac{5}{9} \cdot (86 - 32) = 30°\text{C},$$

$$\sigma_X^2 = \left(\frac{5}{9}\right)^2 \cdot \sigma_T^2 = \left(\frac{5}{9}\right)^2 \cdot 9^2 = 5^2. \qquad \sigma_X = 5°\text{C}.$$

$$P(X > 33) = P\left(Z > \frac{33 - 30}{5}\right) = P(Z > 0.60) = 1 - 0.7257 = \mathbf{0.2743}.$$

8. If the moment-generating function of X is $M(t) = \exp(166\,t + 200\,t^2)$, find

a) The mean of X.

b) The variance of X.

c) $P(170 < X < 200)$.

d) $P(148 < X < 172)$.

$M(t) = e^{166t + 400t^2/2}$ so

(a) $\mu = 166$; **(b)** $\sigma^2 = 400$;

(c) $P(170 < X < 200) = P(0.2 < Z < 1.7) = 0.3761$;

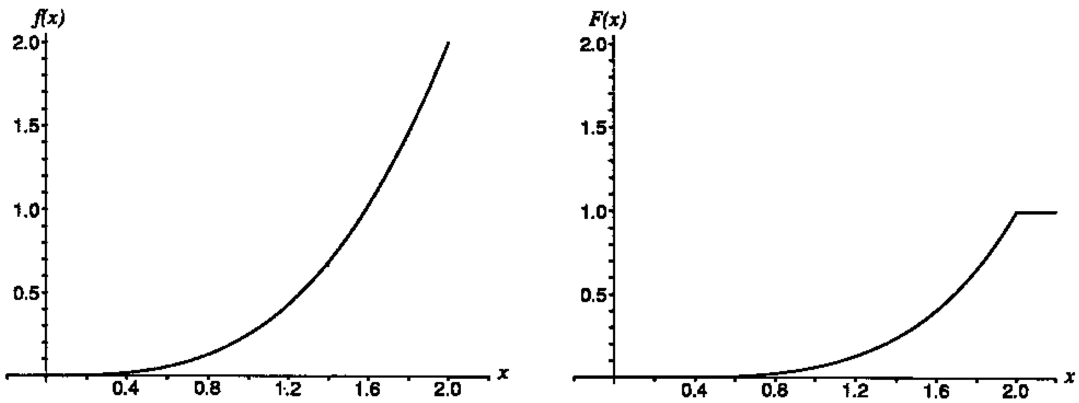
(d) $P(148 \leq X \leq 172) = P(-0.9 \leq Z \leq 0.3) = 0.4338$.

9. **3.1-8 (a)** **3.3-2 (a), 3.3-4 (a)** **3.2-2 (a)**

$$
\begin{aligned}
\text{(a)}\quad \text{(i)}\quad \textstyle\int_0^c x^3/4\,dx &= 1\\
c^4/16 &= 1\\
c &= 2;
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii)}\quad F(x) &= \textstyle\int_{-\infty}^{x} f(t)\,dt\\
&= \textstyle\int_0^x t^3/4\,dt\\
&= x^4/16,
\end{aligned}
$$

$$
F(x) = \begin{cases} 0, & -\infty < x < 0,\\ x^4/16, & 0 \le x < 2,\\ 1, & 2 \le x < \infty. \end{cases}
$$



(a) Continuous distribution p.d.f. and c.d.f.

$$
\begin{aligned}
\text{(a)}\qquad \mu = E(X) &= \int_0^2 \frac{x^4}{4}\,dx\\
&= \left[\frac{x^5}{20}\right]_0^2 = \frac{32}{20} = \frac{8}{5},\\
\sigma^2 = \mathrm{Var}(X) &= \int_0^2 \left(x - \frac{8}{5}\right)^2 \frac{x^3}{4}\,dx\\
&= \int_0^2 \left(\frac{x^5}{4} - \frac{4}{5}x^4 + \frac{16}{25}x^3\right) dx\\
&= \left[\frac{x^6}{24} - \frac{4x^5}{25} + \frac{4x^4}{25}\right]_0^2\\
&= \frac{64}{24} - \frac{128}{25} + \frac{64}{25}\\
&\approx 0.1067,\\
\sigma &= \sqrt{0.1067} = 0.3266;
\end{aligned}
$$

10. **3.1-8 (b)** **3.3-2 (b), 3.3-4 (b)** **3.2-2 (b)**

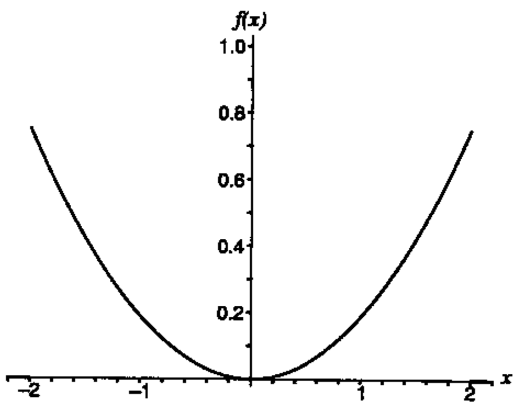
(b) (i)

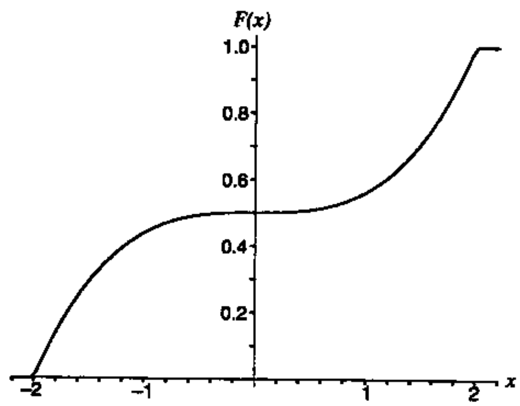
$$\begin{aligned} \int_{-c}^{c}(3/16)x^2\,dx &= 1 \\ c^3/8 &= 1 \\ c &= 2; \end{aligned}$$

(ii)

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\,dt \\ &= \int_{-2}^{x}(3/16)t^2\,dt \\ &= \left[\frac{t^3}{16}\right]_{-2}^{x} \\ &= \frac{x^3}{16}+\frac{1}{2}, \end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < -2, \\ \dfrac{x^3}{16}+\dfrac{1}{2}, & -2 \le x < 2, \\ 1, & 2 \le x < \infty. \end{cases}$$





(b) Continuous distribution p.d.f. and c.d.f.

(b)

$$\begin{aligned} \mu = E(X) &= \int_{-2}^{2}\left(\frac{3}{16}\right)x^3\,dx \\ &= \left[\frac{3}{64}x^4\right]_{-2}^{2} \\ &= \frac{48}{64}-\frac{48}{64}=0, \\ \sigma^2 = \mathrm{Var}(X) &= \int_{-2}^{2}\left(\frac{3}{16}\right)x^4\,dx \\ &= \left[\frac{3}{80}x^5\right]_{-2}^{2} \\ &= \frac{96}{80}+\frac{96}{80} \\ &= \frac{12}{5}, \\ \sigma = \sqrt{\frac{12}{5}} &\approx 1.5492; \end{aligned}$$

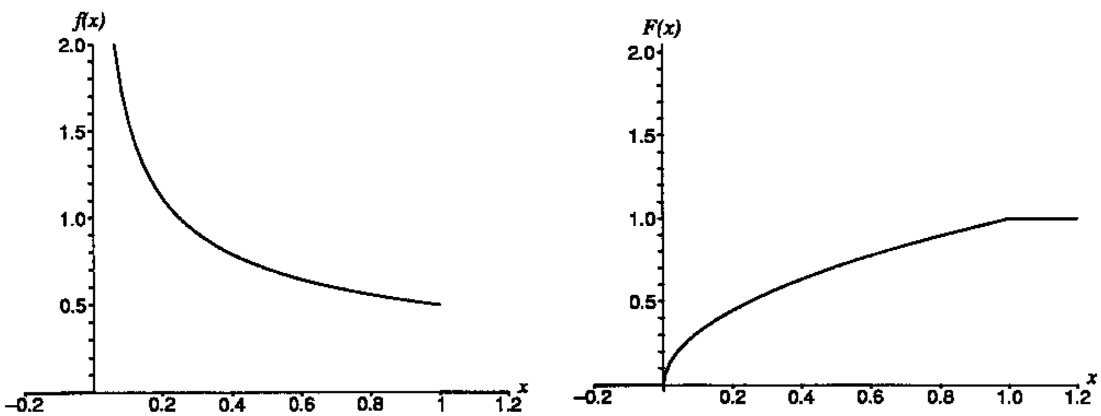
11. **3.1-8 (c)** **3.3-2 (c), 3.3-4 (c)** **3.2-2 (c)**

(c) (i)
$$\begin{aligned} \int_0^1 \frac{c}{\sqrt{x}}\,dx &= 1 \\ 2c &= 1 \\ c &= 1/2. \end{aligned}$$

The p.d.f. in part (c) is unbounded.

(ii)
$$\begin{aligned} F(x) &= \textstyle\int_{-\infty}^{x} f(t)\,dt \\ &= \int_0^x \frac{1}{2\sqrt{t}}\,dt \\ &= \left[\sqrt{t}\right]_0^x = \sqrt{x}, \end{aligned}$$

$$F(x) = \begin{cases} 0, & -\infty < x < 0, \\ \sqrt{x}, & 0 \le x < 1, \\ 1, & 1 \le x < \infty. \end{cases}$$



(c) Continuous distribution p.d.f. and c.d.f.

(c)
$$\begin{aligned} \mu = E(X) &= \int_0^1 \frac{x}{2\sqrt{x}}\,dx \\ &= \int_0^1 \frac{\sqrt{x}}{2}\,dx \\ &= \left[\frac{x^{3/2}}{3}\right]_0^1 = \frac{1}{3}, \\ \sigma^2 = \mathrm{Var}(X) &= \int_0^1 \left(x - \frac{1}{3}\right)^2 \frac{1}{2\sqrt{x}}\,dx \\ &= \int_0^1 \left(\frac{1}{2}x^{3/2} - \frac{2}{6}x^{1/2} + \frac{1}{18}x^{-1/2}\right) dx \\ &= \left[\frac{1}{5}x^{5/2} - \frac{2}{9}x^{3/2} + \frac{1}{9}x^{1/2}\right]_0^1 \\ &= \frac{4}{45}, \\ \sigma = \frac{2}{\sqrt{45}} &\approx 0.2981. \end{aligned}$$

12. **3.1-10** **3.3-8** **3.2-8**

(a)
$$\begin{aligned} \int_1^\infty \frac{c}{x^2}\,dx &= 1 \\ \left[\frac{-c}{x}\right]_1^\infty &= 1 \\ c &= 1; \end{aligned}$$

(b) $E(X) = \displaystyle\int_1^\infty \frac{x}{x^2}\,dx = [\ln x]_1^\infty$, which is unbounded.

13. **3.1-4** **3.4-4** **3.3-4**

X is $U(4,5)$;

(a) $\mu = 9/2$; (b) $\sigma^2 = 1/12$; (c) 0.5.

14. **3.2-2** **3.4-8** **3.3-8**

(a) $f(x) = \left(\dfrac{2}{3}\right) e^{-2x/3}, \qquad 0 \le x < \infty;$

(b) $P(X > 2) = \displaystyle\int_2^\infty \frac{2}{3} e^{-2x/3}\,dx = \left[-e^{-2x/3}\right]_2^\infty = e^{-4/3}.$

15. **3.3-6** **3.6-6** **5.2-6**

$M(t) = e^{166t+400t^2/2}$ so

(a) $\mu = 166$; **(b)** $\sigma^2 = 400$;

(c) $P(170 < X < 200) = P(0.2 < Z < 1.7) = 0.3761$;

(d) $P(148 \leq X \leq 172) = P(-0.9 \leq Z \leq 0.3) = 0.4338$.

16. **3.3-11** **3.6-14** **5.2-14**

(a) $P(X > 22.07) = P(Z > 1.75) = 0.0401$;

(b) $P(X < 20.857) = P(Z < -1.2825) = 0.10$.

Thus the distribution of Y is $b(15,\ 0.10)$
and from Table II in the Appendix, $P(Y \leq 2) = 0.8159$.