Joint Probability Distributions

1. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = C x^2 y^3, \quad 0 < x < 1, \quad 0 < y < \sqrt{x}, \quad \text{zero elsewhere.}$$

a) What must the value of C be so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.?

b) Find $P(X + Y < 1)$.

c) Let $0 < a < 1$. Find $P(Y < aX)$.

d) Let $a > 1$. Find $P(Y < aX)$.

e) Let $0 < a < 1$. Find $P(XY < a)$.

f) Find $f_X(x)$.

g) Find $E(X)$.

h) Find $f_Y(y)$.

i) Find $E(Y)$.

j) Find $E(XY)$.

k) Find $\text{Cov}(X, Y)$.

l) Are X and Y independent?

2. Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{x}, \quad x > 1, \quad 0 < y < \frac{1}{x}, \quad \text{zero elsewhere.}$$

a) Find $f_X(x)$.

b) Find $E(X)$.

c) Find $f_Y(y)$.

d) Find $E(Y)$.

3. Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{x}, \quad 0 < x < 1, \quad 0 < y < x, \quad \text{zero elsewhere.}$$

a) Find $f_X(x)$.

b) Find $E(X)$.

c) Find $f_Y(y)$.

d) Find $E(Y)$.

e) Find $P(X + Y \geq 1)$.

f) Find $\text{Cov}(X, Y)$.

4. Let X and Y be two random variables with joint p.d.f.

$$f(x, y) = 64\,x \exp\{-4y\} = 64\,x e^{-4y}, \qquad 0 < x < y < \infty,$$

zero elsewhere.

a) Find $P(X^2 > Y)$.

b) Find the marginal p.d.f. $f_X(x)$ of X.

c) Find the marginal p.d.f. $f_Y(y)$ of Y.

d) Are X and Y independent? If not, find $Cov(X, Y)$ and $\rho = Corr(X, Y)$.

e) Let $a > 1$. Find $P(Y > aX)$.

f) Let $a > 0$. Find $P(X + Y < a)$.

5. Let the joint probability mass function of X and Y be defined by

$$p(x, y) = \frac{x + y}{32}, \qquad x = 1, 2, \qquad y = 1, 2, 3, 4.$$

a) Find $P(Y > X)$.

b) Find $p_X(x)$, the marginal p.m.f. of X.

c) Find $p_Y(y)$, the marginal p.m.f. of Y.

d) Are X and Y independent? If not, find $Cov(X, Y)$.

6. Let the joint probability mass function of X and Y be defined by

$$p(x, y) = \frac{x \cdot y}{30}, \qquad x = 1, 2, \qquad y = 1, 2, 3, 4.$$

a) Find $P(Y > X)$.

b) Find $p_X(x)$, the marginal p.m.f. of X.

c) Find $p_Y(y)$, the marginal p.m.f. of Y.

d) Are X and Y independent? If not, find $Cov(X, Y)$.

7. Suppose the joint probability density function of (X, Y) is

$$f(x,y)=\begin{cases} C\,x\,y^2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of C that would make $f(x,y)$ a valid probability density function.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

d) Find $P(X > 2Y)$.

e) Find $P(X + Y < 1)$.

f) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

8. Let X and Y have the joint probability density function

$$f(x,y) = Cx, \quad 0 \le x \le 1, \quad 0 \le y \le x(1-x), \qquad \text{zero elsewhere.}$$

a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

b) Find $f_X(x)$.

c) Find $E(X)$.

d) Find $f_Y(y)$.

e) Find $E(Y)$.

f) Are X and Y independent?

9. Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} x + 4y & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find $P(X > 4Y)$.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$.

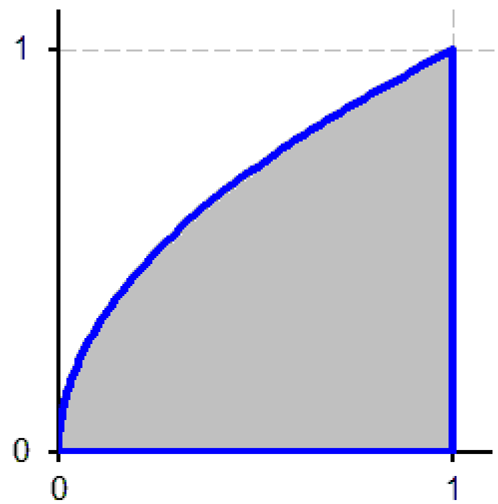
e) Find $P(Y > 0.4 \mid X < 0.8)$.

f) Find $P(X < 0.8 \mid Y > 0.4)$.

1. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = C\,x^2 y^3, \quad 0 < x < 1, \quad 0 < y < \sqrt{x}, \quad \text{zero elsewhere.}$$

a) What must the value of C be so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.?

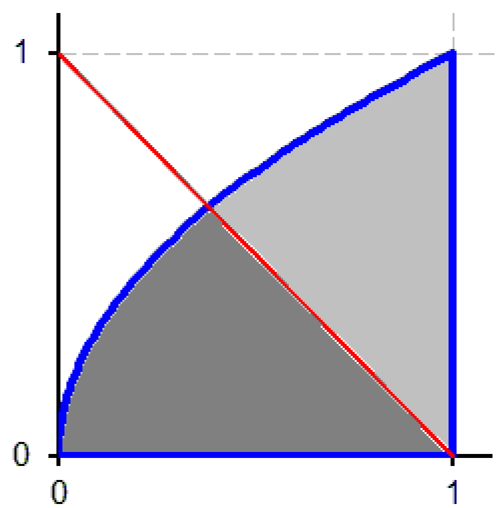


$$\int_0^1 \left(\int_0^{\sqrt{x}} C\,x^2 y^3\,dy \right) dx = \int_0^1 \frac{C}{4} x^4\,dx$$

$$= \frac{C}{20} = 1.$$

$$\Rightarrow \quad C = \mathbf{20}.$$

b) Find $P(X + Y < 1)$.



$y = \sqrt{x}$ and $y = 1 - x$

$x = y^2$ and $x = 1 - y$

$$\Rightarrow \quad y = \frac{\sqrt{5}-1}{2}.$$

$$P(X+Y<1) = \int_0^{\frac{\sqrt{5}-1}{2}} \left(\int_{y^2}^{1-y} 20\,x^2 y^3\,dx \right) dy$$

$$= \int_0^{\frac{\sqrt{5}-1}{2}} \left(\frac{20}{3}(1-y)^3 y^3 - \frac{20}{3} y^9 \right) dy$$

$$= \int_0^{\frac{\sqrt{5}-1}{2}} \left(\frac{20}{3} y^3 - 20\,y^4 + 20\,y^5 - \frac{20}{3} y^6 - \frac{20}{3} y^9 \right) dy$$

$$= \left(\frac{5}{3} y^4 - 4 y^5 + \frac{10}{3} y^6 - \frac{20}{21} y^7 - \frac{2}{3} y^{10} \right) \Bigg|_0^{\frac{\sqrt{5}-1}{2}} \approx 0.030022.$$

OR

$$y < \sqrt{x} \quad \text{and} \quad y = 1 - x \qquad \Rightarrow \qquad x = \left(\frac{\sqrt{5}-1}{2} \right)^2 = 1 - \frac{\sqrt{5}-1}{2} = \frac{3-\sqrt{5}}{2}.$$

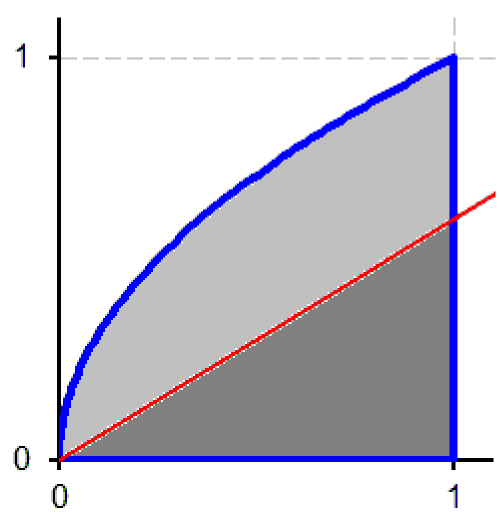
$$\mathrm{P}(\mathrm{X} + \mathrm{Y} < 1) = 1 - \int_{\frac{3-\sqrt{5}}{2}}^{1} \left(\int_{1-x}^{\sqrt{x}} 20 x^2 y^3 \, dy \right) dx$$

$$= 1 - \int_{\frac{3-\sqrt{5}}{2}}^{1} \left(5 x^4 - 5 x^2 (1-x)^4 \right) dx$$

$$= 1 - \int_{\frac{3-\sqrt{5}}{2}}^{1} \left(-5 x^2 + 20 x^3 - 25 x^4 + 20 x^5 - 5 x^6 \right) dy$$

$$= 1 - \left(-\frac{5}{3} x^3 + 5 x^4 - 5 x^5 + \frac{10}{3} x^6 - \frac{5}{7} x^7 \right) \Bigg|_{\frac{3-\sqrt{5}}{2}}^{1} \approx 0.030022.$$

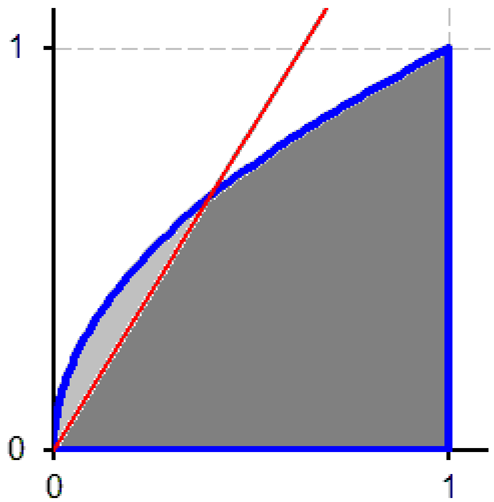
c) Let $0 < a < 1$. Find $\mathrm{P}(\mathrm{Y} < a\,\mathrm{X})$.



$$\mathrm{P}(\mathrm{Y} < a\,\mathrm{X}) = \int_0^1 \left(\int_0^{a x} 20 x^2 y^3 \, dy \right) dx$$

$$= \int_0^1 5 a^4 x^6 \, dx = \frac{5}{7} a^4.$$

d) Let $a > 1$. Find $P(Y < aX)$.



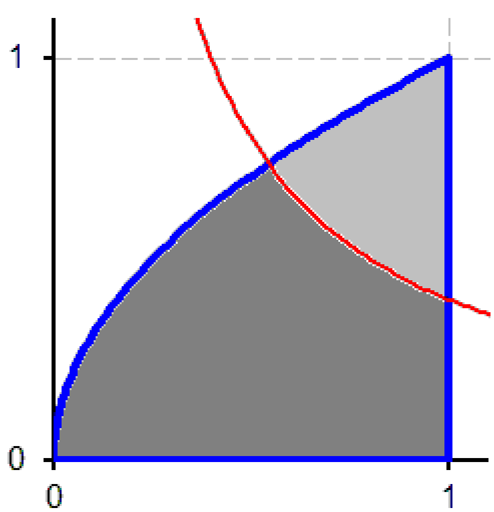
$y = \sqrt{x}$ and $y = a x$

$\Rightarrow \quad x = \frac{1}{a^2}, \qquad y = \frac{1}{a}.$

$$P(Y < aX) = 1 - \int_0^{1/a} \left(\int_{y^2}^{y/a} 20 x^2 y^3 \, dx \right) dy = 1 - \int_0^{1/a} \left(\frac{20 y^6}{3 a^3} - \frac{20}{3} y^9 \right) dy = 1 - \frac{2}{7 a^{10}}.$$

$$P(Y < aX) = 1 - \int_0^{1/a^2} \left(\int_{ax}^{\sqrt{x}} 20 x^2 y^3 \, dy \right) dx = 1 - \int_0^{1/a^2} \left(5 x^4 - 5 a^4 x^6 \right) dx = 1 - \frac{2}{7 a^{10}}.$$

e) Let $0 < a < 1$. Find $P(XY < a)$.



$y = \sqrt{x}$ and $y = \frac{a}{x}$

$\Rightarrow \quad x = a^{2/3}.$

$$P(XY < a) = 1 - \int_{a^{2/3}}^{1} \left(\int_{a/x}^{\sqrt{x}} 20 x^2 y^3 \, dy \right) dx = 1 - \int_{a^{2/3}}^{1} \left(5 x^4 - 5 \frac{a^4}{x^2} \right) dx$$

$$= 1 - \left(x^5 + 5\frac{a^4}{x} \right) \Bigg|_{a^{2/3}}^{1} = 6a^{10/3} - 5a^4.$$

f) Find $f_X(x)$.

$$f_X(x) = \int_0^{\sqrt{x}} 20x^2 y^3 \, dy = 5x^4, \qquad 0 < x < 1.$$

g) Find $E(X)$.

$$E(X) = \int_0^1 x \cdot 5x^4 \, dx = \mathbf{\frac{5}{6}}.$$

h) Find $f_Y(y)$.

$$f_Y(y) = \int_{y^2}^{1} 20x^2 y^3 \, dx = \frac{20}{3} \cdot \left(y^3 - y^9 \right), \qquad 0 < y < 1.$$

i) Find $E(Y)$.

$$E(Y) = \int_0^1 y \cdot \frac{20}{3}\left(y^3 - y^9 \right) dy = \int_0^1 \left(\frac{20}{3} y^4 - \frac{20}{3} y^{10} \right) dy = \frac{4}{3} - \frac{20}{33} = \mathbf{\frac{8}{11}}.$$

j) Find $E(XY)$.

$$E(XY) = \int_0^1 \left(\int_0^{\sqrt{x}} x y \cdot 20x^2 y^3 \, dy \right) dx = \int_0^1 4x^{11/2} \, dx = \mathbf{\frac{8}{13}}.$$

k) Find $\mathrm{Cov}(X, Y)$.

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{8}{13} - \frac{5}{6} \cdot \frac{8}{11} = \mathbf{\frac{8}{858}} \approx 0.009324.$$

l) Are X and Y independent?

$f(x,y) \neq f_X(x) \cdot f_Y(y).$ $\Rightarrow$ X and Y are **NOT independent**.

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

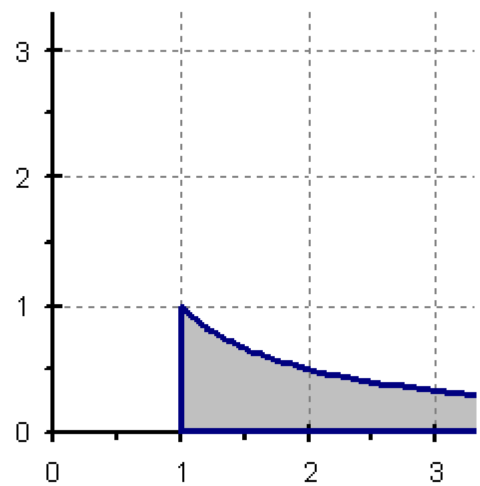
$\text{Cov}(X, Y) \neq 0.$ $\Rightarrow$ X and Y are **NOT independent**.

2. Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{x},$$

$$x > 1, \quad 0 < y < \frac{1}{x},$$

zero elsewhere.



a) Find $f_X(x)$.

$$f_X(x) = \int_0^{1/x} \frac{1}{x}\,dy = \frac{1}{x^2}, \qquad x > 1.$$

b) Find $E(X)$.

Since $\int_1^{\infty} x \cdot \frac{1}{x^2}\,dx = \int_1^{\infty} \frac{1}{x}\,dx = (\ln x)\Big|_1^{\infty}$ diverges, $E(X)$ **does not exist**.

c) Find $f_Y(y)$.

$$f_Y(y) = \int_1^{1/y} \frac{1}{x}\,dx = (\ln x)\Big|_1^{1/y} = \ln\frac{1}{y} - \ln 1 = -\ln y, \qquad 0 < y < 1.$$

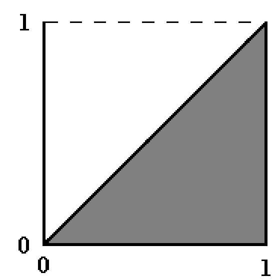
d) Find $E(Y)$.

$$E(Y) = \int_0^1 y(-\ln y)\,dy = \left(-\frac{y^2}{2}\ln y + \frac{y^2}{4}\right)\Bigg|_0^1 = \mathbf{\frac{1}{4}}.$$

3. Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = \frac{1}{x}, \qquad 0 < x < 1, \quad 0 < y < x,$$

zero elsewhere.



a) Find $f_X(x)$.

$$f_X(x) = \int_0^x \frac{1}{x}\,dy = 1, \qquad 0 < x < 1.$$

b) Find $E(X)$.

X has a Uniform distribution on $(0, 1)$. $\Rightarrow$ $E(X) = \mathbf{\frac{1}{2}}$.

OR $$E(X) = \int_0^1 x \cdot 1\,dx = \mathbf{\frac{1}{2}}.$$

c) Find $f_Y(y)$.

$$f_Y(y) = \int_y^1 \frac{1}{x}\,dx = (\ln x)\Big|_y^1 = \ln 1 - \ln y = -\ln y, \qquad 0 < y < 1.$$

d) Find $E(Y)$.

$$E(Y) = \int_0^1 y(-\ln y)\,dy = \left(-\frac{y^2}{2}\ln y + \frac{y^2}{4}\right)\Bigg|_0^1 = \mathbf{\frac{1}{4}}.$$

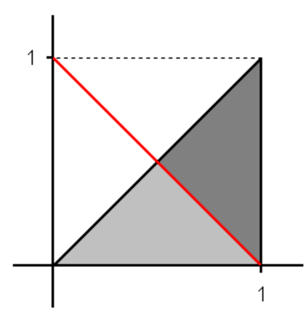
e) Find $P(X+Y \ge 1)$.

$$P(X+Y \ge 1) = \int_{0.5}^1 \left(\int_{1-x}^x \frac{1}{x}\,dy\right) dx$$

$$= \int_{0.5}^1 \frac{2x-1}{x}\,dx = \int_{0.5}^1 \left(2 - \frac{1}{x}\right) dx$$

$$= (2x - \ln x)\Big|_{0.5}^1 = 1 + \ln 0.5$$

$$= \mathbf{1 - \ln 2} \approx 0.3068528.$$



f) Find $\mathrm{Cov}(X,Y)$.

$$E(XY) = \int_0^1 \left(\int_0^x xy \cdot \frac{1}{x}\,dy\right) dx = \int_0^1 \left(\int_0^x y\,dy\right) dx = \int_0^1 \frac{x^2}{2}\,dx = \frac{1}{6}.$$

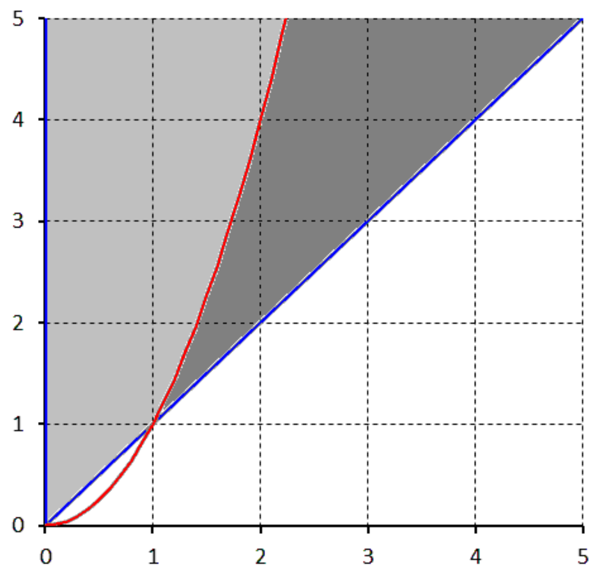
$$\mathrm{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y) = \frac{1}{6} - \frac{1}{2}\cdot\frac{1}{4} = \mathbf{\frac{1}{24}} \approx 0.041667.$$

4. Let X and Y be two random variables with joint p.d.f.

$$f(x, y) = 64\, x \exp\{-4y\} = 64\, x e^{-4y}, \qquad 0 < x < y < \infty,$$

zero elsewhere.

a) Find $P(X^2 > Y)$.



$$P(X^2 > Y) = \int_1^{\infty} \int_x^{x^2} 64\, x e^{-4y}\, dy\, dx$$

$$= \int_1^{\infty} 16\, x e^{-4x}\, dx - \int_1^{\infty} 16\, x e^{-4x^2}\, dx$$

$$u = 4x^2 \qquad du = 8\, x\, dx$$

$$= \left(-4\, x e^{-4x} - e^{-4x}\right)\Big|_1^{\infty} - \int_4^{\infty} 2\, e^{-u}\, du$$

$$= 4\, e^{-4} + e^{-4} - 2\, e^{-4} = \mathbf{3\, e^{-4}} \approx 0.055.$$

b) Find the marginal p.d.f. $f_X(x)$ of X.

$$f_X(x) = \int_x^{\infty} 64\, x e^{-4y}\, dy = 16\, x e^{-4x}, \qquad 0 < x < \infty.$$

X has a Gamma distribution with $\alpha = 2$, $\lambda = 4$.

c) Find the marginal p.d.f. $f_Y(y)$ of Y.

$$f_Y(y) = \int_0^{y} 64\, x e^{-4y}\, dx = 32\, y^2 e^{-4y}, \qquad 0 < y < \infty.$$

Y has a Gamma distribution with $\alpha = 3$, $\lambda = 4$.

d) Are X and Y independent? If not, find $\operatorname{Cov}(X, Y)$ and $\rho = \operatorname{Corr}(X, Y)$.

$f(x, y) \neq f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

OR

The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

X has a Gamma distribution with $\alpha = 2$, $\lambda = 4$. $\quad E(X) = \frac{1}{2}$, $\operatorname{Var}(X) = \frac{1}{8}$.

Y has a Gamma distribution with $\alpha = 3$, $\lambda = 4$. $\quad E(Y) = \frac{3}{4}$, $\operatorname{Var}(Y) = \frac{3}{16}$.

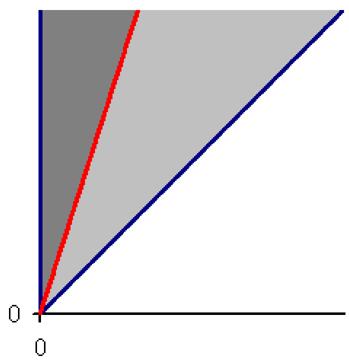
$$E(XY) = \int_0^\infty \int_0^y xy \cdot 64\, x e^{-4y}\, dx\, dy = \int_0^\infty \frac{64}{3} y^4 e^{-4y}\, dy = \int_0^\infty \frac{4^3}{3} y^4 e^{-4y}\, dy$$

$$= \frac{8}{4^2} \cdot \int_0^\infty \frac{4^5}{24} y^4 e^{-4y}\, dy = \frac{1}{2} \cdot \int_0^\infty \frac{4^5}{\Gamma(5)} y^{5-1} e^{-4y}\, dy = \frac{1}{2}.$$

$$\operatorname{Cov}(X, Y) = \frac{1}{2} - \frac{1}{2} \cdot \frac{3}{4} = \mathbf{\frac{1}{8}} = \mathbf{0.125}.$$

$$\rho = \operatorname{Corr}(X, Y) = \frac{1/8}{\sqrt{1/8} \cdot \sqrt{3/16}} = \frac{\sqrt{2}}{\sqrt{3}} \approx 0.8165.$$

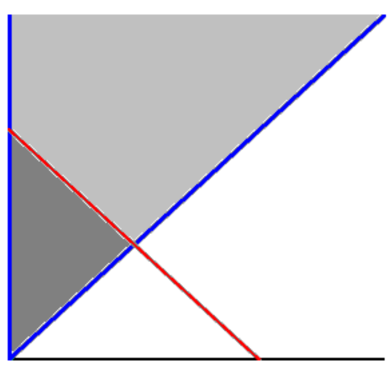
e) Let $a > 1$. Find $\mathrm{P}(\mathrm{Y} > a\,\mathrm{X})$.



$$\mathrm{P}(\mathrm{Y} > a\,\mathrm{X}) = \int_0^\infty \int_{a\,x}^\infty 64\,x\,e^{-4y}\,dy\,dx$$

$$= \int_0^\infty 16\,x\,e^{-4\,a\,x}\,dx = \frac{1}{a^2}.$$

f) Let $a > 0$. Find $\mathrm{P}(\mathrm{X} + \mathrm{Y} < a)$.



$$\mathrm{P}(\mathrm{X} + \mathrm{Y} < a) = \int_0^{a/2} \int_x^{a-x} 64\,x\,e^{-4y}\,dy\,dx$$

$$= \int_1^{a/2} \left(16\,x\,e^{-4x} - 16\,x\,e^{-4a}\,e^{4x}\right) dx$$

$$= \left(-4\,x\,e^{-4x} - e^{-4x} - 4\,x\,e^{-4a}\,e^{4x} + e^{-4a}\,e^{4x}\right)\Big|_0^{a/2}$$

$$= 1 - e^{-4a} - 4\,a\,e^{-2a}.$$

5. Let the joint probability mass function of X and Y be defined by

$$p(x, y) = \frac{x+y}{32}, \qquad x = 1, 2, \qquad y = 1, 2, 3, 4.$$

$x \ \backslash \ y$	1	2	3	4	$p_X(x)$
1	$2/32$	$3/32$	$4/32$	$5/32$	$14/32 = 7/16$
2	$3/32$	$4/32$	$5/32$	$6/32$	$18/32 = 9/16$
$p_Y(y)$	$5/32$	$7/32$	$9/32$	$11/32$	1

a) Find $P(Y > X)$.

$$P(Y > X) = p(1,2) + p(1,3) + p(1,4) + p(2,3) + p(2,4) = 23/32 .$$

b) Find $p_X(x)$, the marginal p.m.f. of X. ↑

c) Find $p_Y(y)$, the marginal p.m.f. of Y. ↑

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

$p(x, y) \neq p_X(x) \cdot p_Y(y)$ X and Y are **NOT independent**.

$$E(X) = 1 \times \frac{7}{16} + 2 \times \frac{9}{16} = \frac{25}{16} = 1.5625.$$

$$E(Y) = 1 \times \frac{5}{32} + 2 \times \frac{7}{32} + 3 \times \frac{9}{32} + 4 \times \frac{11}{32} = \frac{90}{32} = \frac{45}{16} = 2.8125.$$

$$E(XY) = 1 \times \frac{2}{32} + 2 \times \frac{3}{32} + 3 \times \frac{4}{32} + 4 \times \frac{5}{32} + 2 \times \frac{3}{32} + 4 \times \frac{4}{32} + 6 \times \frac{5}{32} + 8 \times \frac{6}{32}$$

$$= \frac{140}{32} = \frac{35}{8} = 4.375.$$

$$\text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{35}{8} - \frac{25}{16} \cdot \frac{45}{16} = -\mathbf{\frac{5}{256}}.$$

6. Let the joint probability mass function of X and Y be defined by

$$p(x,y) = \frac{x \cdot y}{30}, \qquad x = 1, 2, \qquad y = 1, 2, 3, 4.$$

$x \setminus y$	1	2	3	4	$p_X(x)$
1	$1/30$	$2/30$	$3/30$	$4/30$	$10/30 = 1/3$
2	$2/30$	$4/30$	$6/30$	$8/30$	$20/30 = 2/3$
$p_Y(y)$	$1/10$	$2/10$	$3/10$	$4/10$	1

a) Find $P(Y > X)$.

$$P(Y > X) = p(1,2) + p(1,3) + p(1,4) + p(2,3) + p(2,4) = 23/30.$$

b) Find $p_X(x)$, the marginal p.m.f. of X. ↑

c) Find $p_Y(y)$, the marginal p.m.f. of Y. ↑

d) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

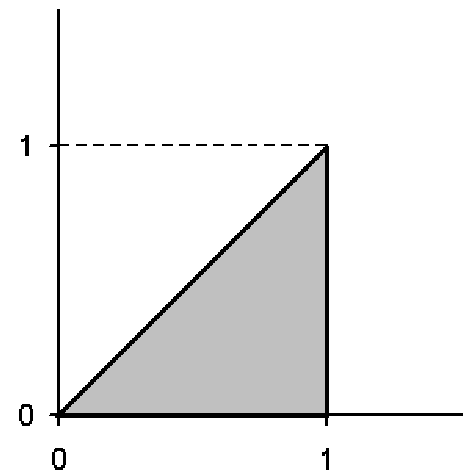
$p(x,y) = p_X(x) \cdot p_Y(y)$ for all x, y.

X and Y are **independent**.

$\text{Cov}(X, Y) = 0.$

7. Suppose the joint probability density function of (X, Y) is

$$f(x,y) = \begin{cases} C\,x\,y^2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



a) Find the value of C that would make $f(x,y)$ a valid probability density function.

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^x C\,x\,y^2\,dy \right) dx = \int_0^1 C\,x \left(\int_0^x y^2\,dy \right) dx$$

$$= \int_0^1 C\,x \left(\frac{1}{3} y^3 \right) \Bigg|_0^x dx = \frac{C}{3} \cdot \int_0^1 x^4\,dx = \frac{C}{3} \cdot \left(\frac{1}{5} x^5 \right) \Bigg|_0^1 = \frac{C}{15}.$$

$\Rightarrow$ $C = \mathbf{15}$.

b) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x 15\,x\,y^2\,dy = 15\,x \left(\int_0^x y^2\,dy \right) = 15\,x \left(\frac{1}{3} y^3 \right) \Bigg|_0^x = 5\,x^4,$$

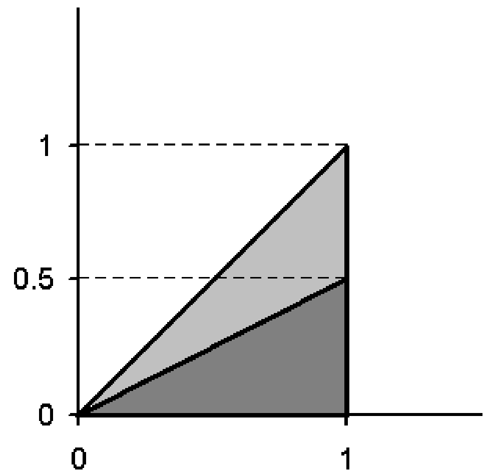
$$0 \le x \le 1.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^1 15\,x\,y^2\,dx = 15\,y^2 \left(\frac{1}{2} x^2 \right) \Bigg|_y^1$$

$$= \frac{15}{2} y^2 \left(1 - y^2 \right) = 7.5\,y^2 - 7.5\,y^4, \qquad 0 \le y \le 1.$$

d) Find $P(X > 2Y)$.



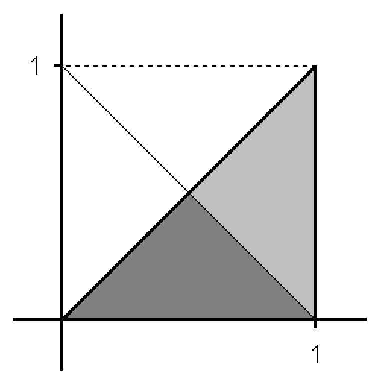
$$P(X > 2Y) = \int_0^1 \left(\int_0^{x/2} 15\,x\,y^2\,dy \right) dx$$

$$= \int_0^1 15\,x \left(\int_0^{x/2} y^2\,dy \right) dx$$

$$= \int_0^1 15\,x \left(\frac{1}{3} y^3 \right) \Bigg|_0^{x/2} dx$$

$$= \int_0^1 \frac{5}{8} x^4\,dx = \frac{1}{8} \cdot \left(x^5 \right) \Big|_0^1 = \mathbf{\frac{1}{8}} = \mathbf{0.125}.$$

e) Find $P(X + Y < 1)$.



$$P(X + Y < 1) = \int_0^{1/2} \left(\int_y^{1-y} 15\,x\,y^2\,dx \right) dy$$

$$= \int_0^{1/2} \frac{15}{2} y^2 \left(\int_y^{1-y} 2\,x\,dx \right) dy$$

$$= \int_0^{1/2} \frac{15}{2} y^2 \left((1-y)^2 - y^2 \right) dy$$

$$= \int_0^{1/2} \frac{15}{2} y^2 (1 - 2y)\,dy = \int_0^{1/2} \left(\frac{15}{2} y^2 - 15\,y^3 \right) dy$$

$$= \left(\frac{5}{2} y^3 - \frac{15}{4} y^4 \right) \Bigg|_0^{1/2} = \frac{5}{16} - \frac{15}{64} = \mathbf{\frac{5}{64}}.$$

f) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

Since the support of (X, Y) is NOT a rectangle, X and Y are **NOT independent**.

OR

Since $f(x, y) \neq f_X(x) \cdot f_Y(y)$, X and Y are **NOT independent**.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x \cdot \left(5x^4\right) dx = \left(\frac{5}{6} \cdot x^6\right)\bigg|_0^1 = \frac{5}{6}.$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^1 y \cdot 15y^2\left(1 - y^2\right) dy = \left(\frac{15}{8} \cdot y^4 - \frac{15}{12} \cdot y^6\right)\bigg|_0^1 = \frac{5}{8}.$$

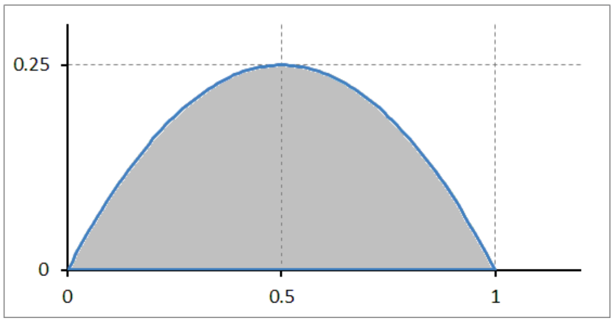
$$E(XY) = \int_0^1 \left(\int_0^x x\,y \cdot 15\,x\,y^2\,dy\right) dx = \int_0^1 \left(\int_0^x \frac{15}{4} x^6\,dy\right) dx = \frac{15}{28}.$$

$$\text{Cov}(X, Y) = E(XY) - E(X) \times E(Y) = \frac{15}{28} - \frac{5}{6} \cdot \frac{5}{8} = \mathbf{\frac{5}{336}} \approx 0.01488.$$

8. Let X and Y have the joint probability density function

$$f(x,y) = Cx, \quad 0 \le x \le 1, \ \ 0 \le y \le x(1-x), \qquad \text{zero elsewhere.}$$

a) Find the value of C so that $f(x,y)$ is a valid joint p.d.f.

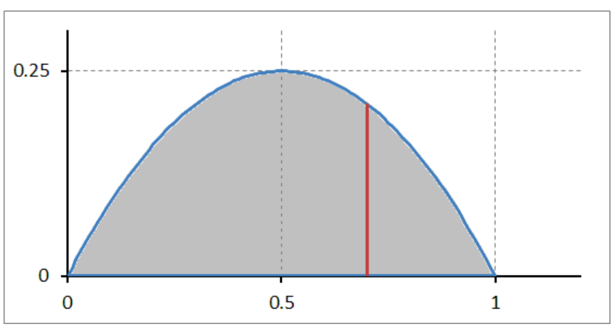


Must have $$1 = \int_0^1 \left(\int_0^{x(1-x)} C\,x\,dy \right) dx$$

$$= C \int_0^1 \left(x^2 - x^3 \right) dx = \frac{C}{12}.$$

$\Rightarrow \qquad C = \mathbf{12}.$

b) Find $f_X(x)$.



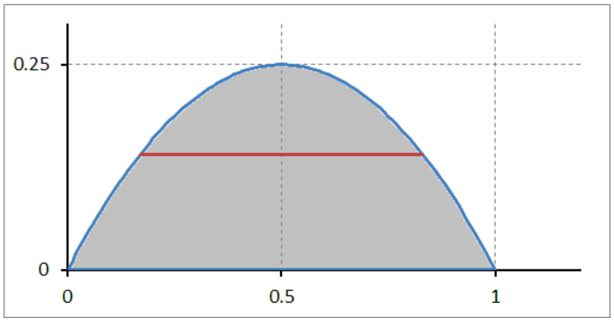
$$f_X(x) = \int_0^{x(1-x)} 12\,x\,dy = 12\,x^2 (1-x),$$

$$0 < x < 1.$$

c) Find $E(X)$.

$$E(X) = \int_0^1 x \cdot 12\,x^2 (1-x)\,dx = \int_0^1 \left(12\,x^3 - 12\,x^4 \right) dx = \left(3\,x^4 - 2.4\,x^5 \right) \Big|_0^1 = \mathbf{0.6}.$$

d) Find $f_Y(y)$.



$y \le x(1-x)$

$\Rightarrow \quad x_1 < x < x_2,$ where

$$x_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - y}, \qquad x_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - y}.$$

$$f_Y(y) = \int_{x_1}^{x_2} 12\,x\,dx = \left(6\,x^2\right)\Big|_{x_1}^{x_2} = 12\sqrt{\frac{1}{4} - y} = 6\sqrt{1-4y}, \qquad 0 < y < \frac{1}{4}.$$

e) Find $E(Y)$.

$$E(Y) = \int_0^{1/4} y \cdot 12\sqrt{\frac{1}{4} - y}\,dy \qquad\qquad u = \frac{1}{4} - y$$

$$= \int_0^{1/4} 12\left(\frac{1}{4} - u\right)\sqrt{u}\,du = \int_0^{1/4}\left(3\,u^{1/2} - 12\,u^{3/2}\right)du$$

$$= \left(2\,u^{3/2} - 4.8\,u^{5/2}\right)\Big|_0^{1/4} = 0.25 - 0.15 = \mathbf{0.10}.$$

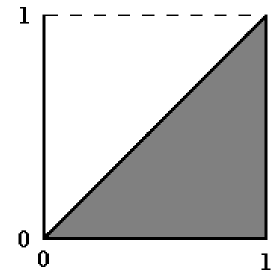
f) Are X and Y independent?

$f(x,y) \ne f_X(x) \cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

OR

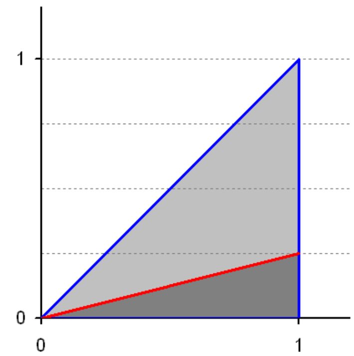
The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

9. Let X and Y have the joint probability density function



$$f_{X,Y}(x,y) = \begin{cases} x+4y & 0<y<x<1 \\ 0 & \text{otherwise} \end{cases}$$

a) Find $P(X>4Y)$.



$$P(X>4Y) = \int_0^1 \left(\int_0^{x/4} (x+4y)\,dy \right) dx$$

$$= \int_0^1 \frac{3x^2}{8}\,dx = \mathbf{\frac{1}{8}}.$$

b) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^x (x+4y)\,dy = \left(xy+2y^2 \right)\Big|_0^x = 3x^2, \qquad 0<x<1.$$

c) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_y^1 (x+4y)\,dx = \left(\frac{x^2}{2}+4xy \right)\Big|_y^1 = \frac{1}{2}+4y-\frac{9}{2}y^2, \qquad 0<y<1.$$

d) Are X and Y independent? If not, find $\text{Cov}(X,Y)$.

The support of (X,Y) is not a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

OR

$f(x,y) \neq f_X(x)\cdot f_Y(y)$. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \int_0^1 x \cdot 3x^2\, dx = \frac{3}{4}.$$

$$E(Y) = \int_0^1 y \cdot \left(\frac{1}{2} + 4y - \frac{9}{2} y^2\right) dy = \frac{1}{4} + \frac{4}{3} - \frac{9}{8} = \frac{11}{24}.$$

$$E(XY) = \int_0^1 \left(\int_0^x x y \cdot (x + 4y)\, dy \right) dx = \int_0^1 \frac{11}{6} x^4\, dx = \frac{11}{30}.$$

$$\mathrm{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{11}{30} - \frac{3}{4} \cdot \frac{11}{24} = \mathbf{\frac{11}{480}}.$$

e) Find $P(Y > 0.4 \mid X < 0.8)$.

Def $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$.

$$P(B) = P(X < 0.8) = \int_0^{0.8} 3x^2\, dx = 0.8^3 = 0.512.$$

$$P(A \cap B) = P(Y > 0.4 \cap X < 0.8) = \int_{0.4}^{0.8} \left(\int_{0.4}^{x} (x + 4y)\, dy \right) dx$$

$$= \int_{0.4}^{0.8} \left(3x^2 - 0.4x - 0.32\right) dx = \left(x^3 - 0.2x^2 - 0.32x\right)\Big|_{0.4}^{0.8} = 0.224.$$

$$P(Y > 0.4 \mid X < 0.8) = \frac{P(A \cap B)}{P(B)} = \frac{0.224}{0.512} = \mathbf{\frac{7}{16}} = \mathbf{0.4375}.$$

f) Find $P(X < 0.8 \mid Y > 0.4)$.

Def $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$, provided $P(A) > 0$.

$$P(A) = P(Y > 0.4) = \int_{0.4}^{1} \left(\frac{1}{2} + 4y - \frac{9}{2} y^2 \right) dy$$

$$= \left(\frac{1}{2} y + 2y^2 - \frac{3}{2} y^3 \right) \Bigg|_{0.4}^{1} = 0.576.$$

$$P(X < 0.8 \mid Y > 0.4) = \frac{P(A \cap B)}{P(A)} = \frac{0.224}{0.576} = \mathbf{\frac{7}{18}} \approx 0.38889.$$