STAT 400 Fall 2017

Moment Generating Functions and Probability Distributions

1. (i) Give the name of the distribution of X (if it has a name), (ii) find the values of μ and σ^2, and (iii) calculate $P(1 \le X \le 2)$ when the moment-generating function of X is given by

a) $M(t) = (0.3 + 0.7e^t)^5$.

b) $M(t) = 0.45 + 0.55e^t$.

c) $M(t) = \dfrac{0.3e^t}{1 - 0.7e^t}$, $t < -\ln(0.7)$.

d) $M(t) = \left(\dfrac{0.6e^t}{1 - 0.4e^t}\right)^2$, $t < -\ln(0.4)$.

e) $M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{4t}$.

f) $M(t) = \sum_{x=1}^{10} (0.1)e^{tx}$.

g) $M(t) = e^{3\left(e^t - 1\right)}$.

h) $M(t) = e^{3t}$.

i) $M_X(t) = \dfrac{1}{1 - 2.5t}$, $t < 0.4$.

j) $M_X(t) = \left(\dfrac{1}{1 - 0.25t}\right)^3$, $t < 4$.

k) $M_X(t) = e^{3t + 2t^2}$.

l) $M_X(t) = \dfrac{e^{5t} - 1}{5t}$, $t \ne 0$, $M_X(0) = 1$.

Answers:

a) $\mathrm{M}(t) = (0.3 + 0.7\,e^{t})^{5}.$

(i) Binomial, $n = 5$, $p = 0.7$.

(ii) $\mu = n\,p = 3.5$, $\sigma^2 = n\,p\,(1 - p) = 1.05$.

(iii) $\binom{5}{1}(0.7)^1(0.3)^4 + \binom{5}{2}(0.7)^2(0.3)^3 = 0.02835 + 0.13230 = 0.16065.$

b) $\mathrm{M}(t) = 0.45 + 0.55\,e^{t}.$

(i) Bernoulli, $p = 0.55$. OR Binomial, $n = 1$, $p = 0.55$.

(ii) $\mu = p = 0.55$, $\sigma^2 = p\,(1 - p) = 0.2475$.

(iii) $0.55 + 0 = 0.55$.

c) $\mathrm{M}(t) = \dfrac{0.3\,e^{t}}{1 - 0.7\,e^{t}}, \qquad t < -\ln(0.7).$

(i) Geometric, $p = 0.3$.

(ii) $\mu = \dfrac{1}{p} = \dfrac{10}{3}, \quad \sigma^2 = \dfrac{1-p}{p^2} = \dfrac{70}{9}.$

(iii) $(0.3) + (0.7)^1(0.3) = 0.51$.

d) $\mathrm{M}(t) = \left(\dfrac{0.6\,e^{t}}{1 - 0.4\,e^{t}}\right)^2, \qquad t < -\ln(0.4).$

(i) Negative Binomial, $p = 0.6$, $r = 2$.

(ii) $\mu = \dfrac{r}{p} = \dfrac{10}{3}, \quad \sigma^2 = \dfrac{r\,(1-p)}{p^2} = \dfrac{20}{9}.$

(iii) $0 + 0.36 = 0.36$.

e) $M(t) = 0.3\,e^{t} + 0.4\,e^{2t} + 0.2\,e^{3t} + 0.1\,e^{4t}$.

(i) No name. $f(1) = 0.3$, $f(2) = 0.4$, $f(3) = 0.2$, $f(4) = 0.1$.

(ii) $\mu = 1 \cdot 0.3 + 2 \cdot 0.4 + 3 \cdot 0.2 + 4 \cdot 0.1 = 2.1$.

$E(X^2) = 1^2 \cdot 0.3 + 2^2 \cdot 0.4 + 3^2 \cdot 0.2 + 4^2 \cdot 0.1 = 5.3$.

$\sigma^2 = 5.3 - 2.1^2 = 0.89$.

(iii) $0.3 + 0.4 = 0.7$.

f) $M(t) = \sum_{x=1}^{10} (0.1) e^{tx}$.

(i) Uniform on integers 1 through 10. $f(1) = f(2) = \ldots = f(10) = 0.1$.

(ii) $\mu = \frac{10+1}{2} = 5.5$, $\sigma^2 = \frac{10^2 - 1}{12} = 8.25$.

(iii) $0.1 + 0.1 = 0.2$.

g) $M(t) = e^{3\left(e^{t} - 1\right)}$.

(i) Poisson, $\lambda = 3$.

(ii) $\mu = \lambda = 3$, $\sigma^2 = \lambda = 3$.

(iii) $P(1 \le X \le 2) = \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} \approx 0.14936 + 0.22404 = 0.37340$.

h) $M(t) = e^{3t}$.

(i) No name. $P(X = 3) = 1$. OR Constant (real number) 3.

(ii) $\mu = 3$, $\sigma^2 = 0$.

(iii) $P(1 \le X \le 2) = 0$.

i) $\quad M_X(t) = \dfrac{1}{1-2.5t}, \quad t < 0.4.$

(i) Exponential, $\theta = 2.5$.

(ii) $E(X) = \theta = 2.5, \qquad \operatorname{Var}(X) = \theta^2 = 6.25.$

(iii) $$P(1 \le X \le 2) = \int_1^2 \frac{1}{2.5} e^{-x/2.5}\,dx = e^{-1/2.5} - e^{-2/2.5}$$
$$= e^{-0.4} - e^{-0.8} \approx 0.220991.$$

j) $\quad M_X(t) = \left(\dfrac{1}{1-0.25t}\right)^3, \quad t < 4.$

(i) Gamma, $\alpha = 3$, $\theta = 0.25$.

(ii) $E(X) = \alpha\,\theta = 0.75, \qquad \operatorname{Var}(X) = \alpha\,\theta^2 = 0.1875.$

(iii) $$P(1 \le X \le 2) = \int_1^2 \frac{1}{\Gamma(3)\cdot 0.25^3} x^{3-1} e^{-x/0.25}\,dx = \int_1^2 32\,x^2 e^{-4x}\,dx$$
$$= \left(-8x^2 e^{-4x} - 4x e^{-4x} - e^{-4x}\right)\Big|_1^2 \approx 0.22435.$$

OR

$$P(1 \le X \le 2) = P(X \ge 1) - P(X \ge 2)$$
$$= P(\text{Poisson}(4) \le 2) - P(\text{Poisson}(8) \le 2)$$
$$\approx 0.238 - 0.014 = 0.224.$$

k) $M_X(t) = e^{3t+2t^2}$.

(i) Normal, $\mu = 3$, $\sigma = 2$.

(ii) $E(X) = \mu = 3$, $\quad$ $Var(X) = \sigma^2 = 4$.

(iii) $P(1 \le X \le 2) = P(-1.00 \le Z \le -0.50) = 0.3085 - 0.1587 = 0.1498$.

l) $M_X(t) = \dfrac{e^{5t} - 1}{5t}$, $\quad t \ne 0$, $\quad$ $M_X(0) = 1$.

(i) Uniform, $a = 0$, $b = 5$.

(ii) $E(X) = \dfrac{a+b}{2} = 2.5$, $\quad$ $Var(X) = \dfrac{(b-a)^2}{12} = \dfrac{25}{12}$.

(iii) $P(1 \le X \le 2) = 0.20$.