STAT 400 Spring 2018

Discussion 03 Solutions

1. A new flavor of toothpaste has been developed. It was tested by a group of 15 people. Nine of the group said they liked the new flavor, and the remaining 6 indicated they did not. Five of the 15 are selected at random to participate in an in-depth interview.

a) What is the probability that of those selected for the in-depth interview 4 liked the new flavor and 1 did not?

$$\frac{{}_9C_4 \cdot {}_6C_1}{{}_{15}C_5} \approx \mathbf{0.2517}.$$

b) What is the probability that of those selected for the in-depth interview at least 4 liked the new flavor?

$$\frac{{}_9C_4 \cdot {}_6C_1}{{}_{15}C_5} + \frac{{}_9C_5 \cdot {}_6C_0}{{}_{15}C_5} \approx \mathbf{0.2937}.$$

2. If a fair 6-sided die is rolled 6 times, what is the probability that each possible outcome (1, 2, 3, 4, 5, and 6) will occur exactly once?

$$\frac{6}{6}\cdot\frac{5}{6}\cdot\frac{4}{6}\cdot\frac{3}{6}\cdot\frac{2}{6}\cdot\frac{1}{6} = \frac{6!}{6^6} = \mathbf{\frac{5}{324}} \approx \mathbf{0.0154321}.$$

3. A laboratory has 15 rats, 7 white, 6 gray and 2 brown. Suppose 5 rats will be selected at random and assigned to an experimental drug.

a) Find the probability that 2 white, 2 gray and 1 brown rats were selected.

$$\begin{array}{ccccc} & & 15 & & \\ & \swarrow & \downarrow & \searrow & \\ 7 & & 6 & & 2 \\ W & & G & & B \\ \downarrow & & \downarrow & & \downarrow \\ 2 & & 2 & & 1 \end{array}$$

$$\frac{{}_7C_2 \cdot {}_6C_2 \cdot {}_2C_1}{{}_{15}C_5} = \frac{21 \cdot 15 \cdot 2}{3003}$$

$$= \frac{630}{3003} \approx \mathbf{0.20979}.$$

b) Find the probability that 3 white and 2 gray rats were selected.

$$\begin{array}{ccccc} & & 15 & & \\ & \swarrow & \downarrow & \searrow & \\ 7 & & 6 & & 2 \\ W & & G & & B \\ \downarrow & & \downarrow & & \downarrow \\ 3 & & 2 & & 0 \end{array}$$

$$\frac{{}_7C_3 \cdot {}_6C_2 \cdot {}_2C_0}{{}_{15}C_5} = \frac{35 \cdot 15 \cdot 1}{3003}$$

$$= \frac{525}{3003} \approx \mathbf{0.174825}.$$

c) Find the probability that all five rats selected have the same color.

P(all five rats have the same color) = P(all five are white) + P(all five are gray)

$$= \frac{{}_7C_5 \cdot {}_6C_0 \cdot {}_2C_0}{{}_{15}C_5} + \frac{{}_7C_0 \cdot {}_6C_5 \cdot {}_2C_0}{{}_{15}C_5} = \frac{21 \cdot 1 \cdot 1}{3003} + \frac{1 \cdot 6 \cdot 1}{3003}$$

$$= \frac{27}{3003} \approx \mathbf{0.008991}.$$

OR

P(all five rats have the same color) = P(all five are white) + P(all five are gray)

$$= \left[\frac{7}{15} \cdot \frac{6}{14} \cdot \frac{5}{13} \cdot \frac{4}{12} \cdot \frac{3}{11}\right] + \left[\frac{6}{15} \cdot \frac{5}{14} \cdot \frac{4}{13} \cdot \frac{3}{12} \cdot \frac{2}{11}\right] = \frac{2520}{360360} + \frac{720}{360360}$$

$$= \frac{3240}{360360} \approx \mathbf{0.008991}.$$

d) Find the probability that all five rats selected have different color.

P(all five rats selected have different color) = $\mathbf{0}$. (3 colors, 5 rats)

4. Does a monkey have a better chance of rearranging

A C C L L S U U to spell C A L C U L U S

or

A A B E G L R to spell A L G E B R A ?

A C C L L S U U $\rightarrow$ C A L C U L U S $\quad \dfrac{8!}{1!\cdot 2!\cdot 2!\cdot 1!\cdot 2!} = 5040$ ways.

A A B E G L R $\rightarrow$ A L G E B R A $\quad \dfrac{7!}{2!\cdot 1!\cdot 1!\cdot 1!\cdot 1!\cdot 1!} = 2520$ ways.

Spelling A L G E B R A is twice as likely as spelling C A L C U L U S.

5. **2.1-10** **2.1-10**

Suppose there are 3 defective items in a lot (collection) of 50 items. A sample of size 10 is taken at random and without replacement. Let X denote the number of defective items in the sample. Find the probability that the sample contains

a) Exactly one defective item.

$$\frac{\binom{3}{1}\cdot\binom{47}{9}}{\binom{50}{10}} = \frac{\frac{3!}{1!\cdot 2!}\cdot\frac{47!}{9!\cdot 38!}}{\frac{50!}{10!\cdot 40!}} = \frac{3\cdot 10\cdot 40\cdot 39}{50\cdot 49\cdot 48} = \frac{39}{98} \approx 0.39796.$$

b) At most one defective item.

$$\frac{\binom{3}{0}\cdot\binom{47}{10}}{\binom{50}{10}} + \frac{\binom{3}{1}\cdot\binom{47}{9}}{\binom{50}{10}} = \frac{1\cdot\frac{47!}{10!\cdot 37!}}{\frac{50!}{10!\cdot 40!}} + \frac{39}{98} = \frac{40\cdot 39\cdot 38}{50\cdot 49\cdot 48} + \frac{39}{98}$$

$$= \frac{247}{490} + \frac{195}{490} = \frac{221}{245} \approx 0.90204.$$

6 – 7. Sally sells seashells by the seashore. The daily sales X of the seashells have the following probability distribution:

x	$f(x)$	$x \cdot f(x)$	$(x-\mu)^2 \cdot f(x)$	$x^2 \cdot f(x)$
0	*0.30*	0.00	0.6750	0.00
1	0.25	0.25	0.0625	0.25
2	0.20	0.40	0.0500	0.80
3	0.15	0.45	0.3375	1.35
4	0.10	0.40	0.6250	1.60
	1.00	**1.50**	**1.7500**	4.00

6. a) Find the missing probability $f(0) = \mathrm{P}(\mathrm{X}=0)$.

$f(0) = 1 - [0.25 + 0.20 + 0.15 + 0.10] = 1 - 0.70 = \mathbf{0.30}$.

b) Find the probability that she sells at least three seashells on a given day.

$\mathrm{P}(\mathrm{X} \ge 3) = \mathrm{P}(\mathrm{X} = 3) + \mathrm{P}(\mathrm{X} = 4) = 0.15 + 0.10 = \mathbf{0.25}$.

c) Find the expected daily sales of seashells.

$$\mathrm{E(X)} = \mu_{\mathrm{X}} = \sum_{\text{all } x} x \cdot f(x) = \mathbf{1.50}.$$

d) Find the standard deviation of daily sales of seashells.

$$\sigma_{\mathrm{X}}^2 = \mathrm{Var(X)} = \sum_{\text{all } x} (x-\mu)^2 \cdot f(x) = \mathbf{1.75}.$$

OR

$$\sigma_{\mathrm{X}}^2 = \mathrm{Var(X)} = \sum_{\text{all } x} x^2 \cdot f(x) - [\,\mathrm{E(X)}\,]^2 = 4.00 - (1.50)^2 = 4.00 - 2.25 = \mathbf{1.75}.$$

$$\sigma_{\mathrm{X}} = \mathrm{SD(X)} = \sqrt{1.75} = \mathbf{1.3229}.$$

7. Suppose each shell sells for \$5.00. However, Sally must pay \$3.00 daily for the permit to sell shells. Let Y denote Sally's daily profit. Then $\mathbf{Y = 5 \cdot X - 3}$.

e) Find the probability that Sally's daily profit will be at least \$10.00.

x	y	$f(x) = f(y)$
0	–\$3.00	0.30
1	\$2.00	0.25
2	\$7.00	0.20
3	\$12.00	0.15
4	\$17.00	0.10
		1.00

$P(Y \geq 10) = P(X \geq 3) =$ **0.25**.

(Sally's daily profit is at least \$10.00 if and only if she sells at least 3 shells.)

f) Find Sally's expected daily profit.

$\mu_Y = E(Y) = \$5.00 \cdot E(X) - \$3.00 = \$5.00 \cdot 1.50 - \$3.00 = \mathbf{\$4.50}$.

(On average, Sally sells 1.5 shells per day, her expected revenue is \$7.50. Her expected profit is \$4.50 since she has to pay \$3.00 for the permit.)

OR

x	y	$f(x) = f(y)$	$y \cdot f(y)$
0	–\$3.00	0.30	–0.90
1	\$2.00	0.25	0.50
2	\$7.00	0.20	1.40
3	\$12.00	0.15	1.80
4	\$17.00	0.10	1.70
		1.00	**4.50**

$$\mu_Y = E(Y) = \sum_{\text{all } y} y \cdot f(y) = \mathbf{\$4.50}.$$

g) Find the standard deviation of Sally's daily profit.

$\sigma_Y = SD(Y) = |5| \cdot SD(X) = 5 \cdot 1.3229 = \$\mathbf{6.6145}$.

OR

x	y	$f(x) = f(y)$	$(y - \mu_Y)^2 \cdot f(y)$	$y^2 \cdot f(y)$
0	−\$3.00	0.30	16.8750	2.70
1	\$2.00	0.25	1.5625	1.00
2	\$7.00	0.20	1.2500	9.80
3	\$12.00	0.15	8.4375	21.60
4	\$17.00	0.10	15.6250	28.90
		1.00	**43.7500**	64.00

$$\sigma_Y^2 = \text{Var}(Y) = \sum_{\text{all } y} (y - \mu_Y)^2 \cdot f(y) = \mathbf{43.75}.$$

OR

$$\sigma_Y^2 = \text{Var}(Y) = \sum_{\text{all } y} y^2 \cdot f(y) - [\,E(Y)\,]^2 = 64.00 - (4.50)^2$$

$$= 64.00 - 20.25 = \mathbf{43.75}.$$

$\sigma_Y = SD(Y) = \sqrt{43.75} = \$\mathbf{6.6144}$.

8. How much wood would a woodchuck chuck if a woodchuck could chuck wood? Let X denote the amount of wood a woodchuck would chuck per day (in cubic meters) if a woodchuck could chuck wood. Find the missing probability $f(0) = \mathrm{P}(\mathrm{X} = 0)$, the average amount of wood a woodchuck would chuck per day, $\mathrm{E}(\mathrm{X})$, and the variance $\mathrm{Var}(\mathrm{X})$.

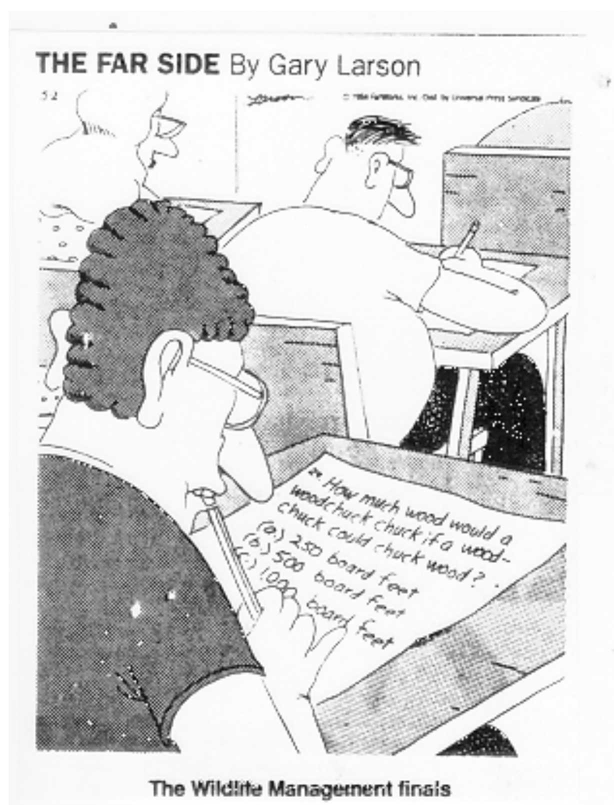
x	$f(x)$	$x \cdot f(x)$	$x^2 \cdot f(x)$
0		0.0	0.0
5	0.30	1.5	7.5
10	0.40	4.0	40.0
15	0.10	1.5	22.5
		7.0	70.0

$f(0) = 1 - [\,0.30 + 0.40 + 0.10\,] = \mathbf{0.20}$.

$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot f(x) = \mathbf{7}$.

$\mathrm{E}(\mathrm{X}^2) = \sum_{\text{all } x} x^2 \cdot f(x) = 70$.

$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = 70 - 7^2 = \mathbf{21}$.



The Wildlife Management finals

9. *Initech* does part of its business via Internet. However, because of the high traffic, the server that carries the company's website often crashes. The probability distribution of the number of server crashes per month, X, is given below:

x	$f(x)$	$x \cdot f(x)$	$x^2 \cdot f(x)$
0	0.15	0	0
1	0.35	0.35	0.35
2	0.45	0.90	1.80
3	0.05	0.15	0.45
		1.40	2.60

If the server does not crash, *Initech*'s monthly profit from the website is $80,000. However, each time the server crashes, *Initech* loses $30,000. Therefore, *Initech*'s monthly profit from the website (in thousands of $) is $Y = 80 - 30 \cdot X$. Find *Initech*'s average monthly profit from the website and its standard deviation.

$$E(X) = \sum_{\text{all } x} x \cdot f(x) = \mathbf{1.40}.$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \sum_{\text{all } x} x^2 \cdot f(x) - [E(X)]^2 = 2.60 - (1.40)^2 = \mathbf{0.64}.$$

$$SD(X) = \sqrt{0.64} = \mathbf{0.80}.$$

$$E(Y) = 80 - 30 \cdot E(X) = 80 - 30 \cdot 1.40 = \mathbf{38}. \qquad \mathbf{\$38{,}000}.$$

$$SD(Y) = |-30| \cdot SD(X) = 30 \cdot 0.80 = \mathbf{24}. \qquad \mathbf{\$24{,}000}.$$

10. Consider $f(x) = c\left(\frac{1}{4}\right)^x$, $x = 1, 2, 3, 4, \ldots.$

a) Find c such that $f(x)$ satisfies the conditions of being a p.m.f. for a random variable X.

$$1 = \sum_{\text{all } x} f(x) = f(1) + f(2) + f(3) + f(4) + \ldots = \frac{c}{4} + \frac{c}{4^2} + \frac{c}{4^3} + \frac{c}{4^4} + \ldots$$

$$= \frac{\textit{first term}}{1 - \textit{base}} = \frac{\frac{c}{4}}{1 - \frac{1}{4}} = \frac{c}{3}.$$

$\Rightarrow \quad c = \mathbf{3}.$

b) Find the expected value of X.

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot f(x) = 1 \times \frac{3}{4} + 2 \times \frac{3}{4^2} + 3 \times \frac{3}{4^3} + 4 \times \frac{3}{4^4} + \ldots .$$

$$\frac{1}{4} \times \mathrm{E}(\mathrm{X}) = 1 \times \frac{3}{4^2} + 2 \times \frac{3}{4^3} + 3 \times \frac{3}{4^4} + \ldots .$$

$$\Rightarrow \quad \frac{3}{4} \times \mathrm{E}(\mathrm{X}) = 1 \times \frac{3}{4} + 1 \times \frac{3}{4^2} + 1 \times \frac{3}{4^3} + 1 \times \frac{3}{4^4} + \ldots = 1.$$

$$\Rightarrow \quad \mathrm{E}(\mathrm{X}) = \mathbf{\frac{4}{3}} \approx 1.33333.$$

OR

Geometric distribution with $p = \frac{3}{4}$.

$$\Rightarrow \quad \mathrm{E}(\mathrm{X}) = \frac{1}{p} = \mathbf{\frac{4}{3}} \approx 1.33333.$$

c) Find $P(X \text{ is odd})$.

$$P(X \text{ is odd}) = f(1)+f(3)+f(5)+f(7)+\ldots = \frac{3}{4}+\frac{3}{4^3}+\frac{3}{4^5}+\frac{3}{4^7}+\ldots$$

$$= \frac{\textit{first term}}{1-\textit{base}} = \frac{\frac{3}{4}}{1-\frac{1}{4^2}} = \frac{12}{15} = \mathbf{\frac{4}{5}} = \mathbf{0.80}.$$

OR

$$P(X \text{ is odd}) = f(1)+f(3)+f(5)+f(7)+\ldots = \frac{3}{4}+\frac{3}{4^3}+\frac{3}{4^5}+\frac{3}{4^7}+\ldots .$$

$$P(X \text{ is even}) = f(2)+f(4)+f(6)+f(8)+\ldots = \frac{3}{4^2}+\frac{3}{4^4}+\frac{3}{4^6}+\frac{3}{4^8}+\ldots$$

$$= \frac{1}{4} \times P(X \text{ is odd}).$$

$P(X \text{ is odd}) + P(X \text{ is even}) = 1.$ $\qquad \frac{5}{4} \times P(X \text{ is odd}) = 1.$

$$\Rightarrow \quad P(X \text{ is odd}) = \mathbf{\frac{4}{5}} = \mathbf{0.80}.$$

11. Find $\mathrm{E}(\mathrm{X})$ for the following discrete probability distributions:

a) $f(0) = \frac{7}{8}$, $f(k) = \frac{1}{3^k}$, $k = 2, 4, 6, 8, \ldots$.

(possible values of X are even non-negative integers: 0, 2, 4, 6, 8, …).

Recall Week 02 Discussion Problem 1 (a): this is a valid probability distribution.

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot p(x) = 0 \cdot \frac{7}{8} + \frac{2}{3^2} + \frac{4}{3^4} + \frac{6}{3^6} + \frac{8}{3^8} + \ldots$$

$$\frac{1}{9}\,\mathrm{E}(\mathrm{X}) = \frac{2}{3^4} + \frac{4}{3^6} + \frac{6}{3^8} + \ldots$$

$$\Rightarrow \quad \frac{8}{9}\,\mathrm{E}(\mathrm{X}) = \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \frac{2}{3^8} + \ldots = \frac{\frac{2}{9}}{1 - \frac{1}{9}} = \frac{1}{4}.$$

$$\Rightarrow \quad \mathrm{E}(\mathrm{X}) = \mathbf{\frac{9}{32}}.$$

OR

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot p(x) = 0 \cdot \frac{7}{8} + \sum_{k=1}^{\infty} 2k \cdot \frac{1}{3^{2k}} = 2 \cdot \sum_{k=1}^{\infty} k \cdot \frac{1}{9^k}$$

$$= \frac{2}{8} \cdot \sum_{k=1}^{\infty} k \cdot \left(\frac{1}{9}\right)^{k-1} \cdot \frac{8}{9} = \frac{2}{8} \cdot \mathrm{E}(\mathrm{Y}),$$

where Y has a Geometric distribution with probability of "success" $p = \frac{8}{9}$.

$$\Rightarrow \quad \mathrm{E}(\mathrm{X}) = \frac{2}{8} \cdot \mathrm{E}(\mathrm{Y}) = \frac{2}{8} \cdot \frac{9}{8} = \mathbf{\frac{9}{32}}.$$

b) $f(1) = \ln 3 - 1,$ $\qquad f(k) = \dfrac{(\ln 3)^k}{k!}, \quad k = 2, 3, 4, \ldots .$

(possible values of X are positive integers: 1, 2, 3, 4, …).

Recall Week 02 Discussion Problem 1 (b): this is a valid probability distribution.

$$\mathrm{E}(\mathrm{X}) = \sum_{\text{all } x} x \cdot p(x) = 1 \cdot (\ln 3 - 1) + \sum_{k=2}^{\infty} k \cdot \frac{(\ln 3)^k}{k!}$$

$$= \ln 3 - 1 + \sum_{k=2}^{\infty} \frac{(\ln 3)^k}{(k-1)!} = \ln 3 - 1 + (\ln 3) \cdot \sum_{k=2}^{\infty} \frac{(\ln 3)^{k-1}}{(k-1)!}$$

$$= \ln 3 - 1 + (\ln 3) \cdot \sum_{k=1}^{\infty} \frac{(\ln 3)^k}{k!} = \ln 3 - 1 + (\ln 3) \cdot \left(e^{\ln 3} - 1\right)$$

$$= \mathbf{3 \ln 3 - 1} \approx 2.2958.$$

12. Each of three balls is randomly placed into one of three bowls. Let X denote the number of empty bowls.

a) Find the probability distribution of X.

"Hint": $f(0) = \mathrm{P}(\mathrm{X} = 0)$ and $f(2) = \mathrm{P}(\mathrm{X} = 2)$ are easier to find.

$$f(0) = \mathrm{P}(\mathrm{X}=0) \;=\; \mathrm{P}(\text{no empty bowls}) \;=\; \mathrm{P}(\text{one ball in each bowl}) \;=\; \frac{3}{3}\times\frac{2}{3}\times\frac{1}{3} \;=\; \frac{2}{9}.$$

$$f(2) = \mathrm{P}(\mathrm{X}=2) \;=\; \mathrm{P}(\text{two empty bowls}) \;=\; \mathrm{P}(\text{all balls in one bowl}) \;=\; \frac{3}{3}\times\frac{1}{3}\times\frac{1}{3} \;=\; \frac{1}{9}.$$

$$f(1) = \mathrm{P}(\mathrm{X}=1) \;=\; 1 - f(0) - f(2) \;=\; 1 - \frac{2}{9} - \frac{1}{9} \;=\; \frac{6}{9}.$$

OR

1 2 3				1 2 3				1 2 3
1 2	3		3	1 2		3		1 2
1 2		3		1 2	3		3	1 2
1 3	2		2	1 3		2		1 3
1 3		2		1 3	2		2	1 3
1	2 3		2 3	1		2 3		1
1		2 3		1	2 3		2 3	1
1	2	3	2	1	3	2	3	1
1	3	2	3	1	2	3	2	1

b) Find $\mathrm{E}(\mathrm{X})$.

$$\mathrm{E}(\mathrm{X}) \;=\; \sum_{\text{all } x} x \cdot f(x) \;=\; 0\cdot\frac{2}{9} + 1\cdot\frac{6}{9} + 2\cdot\frac{1}{9} \;=\; \mathbf{\frac{8}{9}}$$

13. Let the random variable X be the number of days that a certain patient needs to be in the hospital. Suppose X has the p.m.f.

$$f(x) = \frac{5-x}{10}, \qquad x = 1,2,3,4.$$

a) Find $\mathrm{E}(\mathrm{X})$, the expected number of days the patient needs to be in the hospital.

x	$f(x)$	$x \cdot f(x)$	
1	0.40	0.40	
2	0.30	0.60	
3	0.20	0.60	
4	0.10	0.40	
		2.00	$= \mathrm{E}(\mathrm{X})$

b) **2.2-3** **2.2-5**

If the patient is to receive \$200 from the insurance company for each of the first two days in the hospital and \$100 for each day after the first two days, what is the expected payment for the hospitalization?

"Hint": If the patient spends three days in the hospital, the payment is \$500.

x	payment	$f(x)$	payment $\cdot f(x)$	
1	\$200	0.40	80	
2	\$400	0.30	120	
3	\$500	0.20	100	
4	\$600	0.10	60	
			\$360	$= \mathrm{E}(\text{payment})$

14. Suppose we roll a pair of fair 6-sided dice. Let X denote the sum of the outcomes on the two dice. Construct the probability distribution of X and compute its expected value.

(1 , 1)	(1 , 2)	(1 , 3)	(1 , 4)	(1 , 5)	(1 , 6)
(2 , 1)	(2 , 2)	(2 , 3)	(2 , 4)	(2 , 5)	(2 , 6)
(3 , 1)	(3 , 2)	(3 , 3)	(3 , 4)	(3 , 5)	(3 , 6)
(4 , 1)	(4 , 2)	(4 , 3)	(4 , 4)	(4 , 5)	(4 , 6)
(5 , 1)	(5 , 2)	(5 , 3)	(5 , 4)	(5 , 5)	(5 , 6)
(6 , 1)	(6 , 2)	(6 , 3)	(6 , 4)	(6 , 5)	(6 , 6)

x	$f(x)$
2	$^1/_{36}$
3	$^2/_{36}$
4	$^3/_{36}$
5	$^4/_{36}$
6	$^5/_{36}$
7	$^6/_{36}$
8	$^5/_{36}$
9	$^4/_{36}$
10	$^3/_{36}$
11	$^2/_{36}$
12	$^1/_{36}$

$E(X) = \mathbf{7}$.

15. Suppose we roll a pair of fair 6-sided dice. Let Y denote the maximum (the largest) of the outcomes on the two dice. Construct the probability distribution of Y and compute its expected value.

(1 , 1)	(1 , 2)	(1 , 3)	(1 , 4)	(1 , 5)	(1 , 6)
(2 , 1)	(2 , 2)	(2 , 3)	(2 , 4)	(2 , 5)	(2 , 6)
(3 , 1)	(3 , 2)	(3 , 3)	(3 , 4)	(3 , 5)	(3 , 6)
(4 , 1)	(4 , 2)	(4 , 3)	(4 , 4)	(4 , 5)	(4 , 6)
(5 , 1)	(5 , 2)	(5 , 3)	(5 , 4)	(5 , 5)	(5 , 6)
(6 , 1)	(6 , 2)	(6 , 3)	(6 , 4)	(6 , 5)	(6 , 6)

y	$f(y)$
1	$^{1}/_{36}$
2	$^{3}/_{36}$
3	$^{5}/_{36}$
4	$^{7}/_{36}$
5	$^{9}/_{36}$
6	$^{11}/_{36}$

$y \times f(y)$
$^{1}/_{36}$
$^{6}/_{36}$
$^{15}/_{36}$
$^{28}/_{36}$
$^{45}/_{36}$
$^{66}/_{36}$

$$E(Y) = {}^{161}/_{36} \approx 4.472222.$$

16. Suppose we roll a pair of fair 6-sided dice. Let Z denote the difference between the outcomes on the two dice. Construct the probability distribution of Z and compute its expected value.

(1 , 1)	(1 , 2)	(1 , 3)	(1 , 4)	(1 , 5)	(1 , 6)
(2 , 1)	(2 , 2)	(2 , 3)	(2 , 4)	(2 , 5)	(2 , 6)
(3 , 1)	(3 , 2)	(3 , 3)	(3 , 4)	(3 , 5)	(3 , 6)
(4 , 1)	(4 , 2)	(4 , 3)	(4 , 4)	(4 , 5)	(4 , 6)
(5 , 1)	(5 , 2)	(5 , 3)	(5 , 4)	(5 , 5)	(5 , 6)
(6 , 1)	(6 , 2)	(6 , 3)	(6 , 4)	(6 , 5)	(6 , 6)

z	$f(z)$
0	$^{6}/_{36}$
1	$^{10}/_{36}$
2	$^{8}/_{36}$
3	$^{6}/_{36}$
4	$^{4}/_{36}$
5	$^{2}/_{36}$

$z \times f(z)$
0
$^{10}/_{36}$
$^{16}/_{36}$
$^{18}/_{36}$
$^{16}/_{36}$
$^{10}/_{36}$

$$E(Z) = {}^{70}/_{36} = {}^{35}/_{18} \approx 1.944444.$$

17. The birthday problem. A probability class has N students.

a) What is the probability that at least 2 students in the class have the same birthday? For simplicity, assume that there are always 365 days in a year and that birth rates are constant throughout the year.

$$1 - P(\text{all } N \text{ students have birthdays on different days})$$

$$= 1 - \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{365 - N + 1}{365} = 1 - \frac{{}_{365}P_N}{365^N}.$$

b) Use a computer of a calculator to find the smallest class size for which the probability that at least 2 students in the class have the same birthday exceeds 0.5.

1	0 (obviously)
2	0.00274
3	0.008204
4	0.016356
5	0.027136
6	0.040462
7	0.056236
8	0.074335
9	0.094624
10	0.116948
11	0.141141
12	0.167025
13	0.19441
14	0.223103
15	0.252901

16	0.283604
17	0.315008
18	0.346911
19	0.379119
20	0.411438
21	0.443688
22	0.475695
23	**0.507297**
24	0.538344
25	0.5687
26	0.598241
27	0.626859
28	0.654461
29	0.680969
30	0.706316

18. a) How many ways are there to rearrange letters in the word **STATISTICS**?

10 letters: 1 **A**, 1 **C**, 2 **I**, 3 **S**, 3 **T**.

$$\frac{10!}{1!\cdot 1!\cdot 2!\cdot 3!\cdot 3!} = \mathbf{50{,}400}.$$

OR

$${}_{10}C_1 \cdot {}_{9}C_1 \cdot {}_{8}C_2 \cdot {}_{6}C_3 \cdot {}_{3}C_3 = 10 \cdot 9 \cdot 28 \cdot 20 \cdot 1 = \mathbf{50{,}400}.$$

b) If three letters are selected at random from the word **STATISTICS**, what is the probability that the selected letters can be rearranged to spell **CAT**? ("Hint": The order of the selection is NOT important, the selection is done without replacement.) That is, what is the probability that selected letters are **A**, **C**, and **T**?

$$\frac{{}_1C_1 \cdot {}_1C_1 \cdot {}_2C_0 \cdot {}_3C_0 \cdot {}_3C_1}{{}_{10}C_3} = \frac{1\cdot 1\cdot 1\cdot 1\cdot 3}{120} = \mathbf{\frac{1}{40}} = \mathbf{0.025}.$$

c) If six letters are selected at random from the word **STATISTICS**, what is the probability that the selected letters can be rearranged to spell **ASSIST**?

$$\frac{{}_1C_1 \cdot {}_1C_0 \cdot {}_2C_1 \cdot {}_3C_3 \cdot {}_3C_1}{{}_{10}C_6} = \frac{1\cdot 1\cdot 2\cdot 1\cdot 3}{210} = \mathbf{\frac{1}{35}} \approx 0.02857.$$

19. Honest Harry introduced a new "savings opportunity" at *Honest Harry's Used Car Dealership*. A customer is offered the opportunity to roll four fair (balanced) 6-sided dice. If the customer rolls at least one "6", Honest Harry takes an extra \$1,000 off the price of the car. However, if the customer does not have a "6", an extra \$1,000 is added to the price of the car. What is the customer's expected savings from accepting this "savings opportunity"?

P(no "6") = $\left(\frac{5}{6}\right)\cdot\left(\frac{5}{6}\right)\cdot\left(\frac{5}{6}\right)\cdot\left(\frac{5}{6}\right)=\frac{625}{1296}\approx 0.48225.$

P(at least one "6") = 1 – P(no "6") = $1-\frac{625}{1296}=\frac{671}{1296}\approx 0.51775.$

Let X denote the amount saved by the customer. Then X has the following probability distribution:

x	$p(x)$	$x\cdot p(x)$
$-1{,}000$	$\frac{625}{1296}$	-482.25
$1{,}000$	$\frac{671}{1296}$	517.75
		35.50

E(X) ≈ **\$35.50**.

20. A box contains 2 green and 3 red marbles. A person draws a marble from the box at random **without replacement**. If the marble drawn is red, the game stops. If it is green, the person draws again until the red marble is drawn (note that total number of marbles drawn cannot exceed 3 since there are only 2 green marbles in the box at the start). Let the random variable X denote the number of **green** marbles drawn.

a) Find the probability distribution of X. [Hint: There are only 3 possible outcomes for this experiment. It would be helpful to list these three outcomes (three possible sequences of colors of marbles drawn). It may be helpful to make a tree diagram for this experiment. Remember that all (three) probabilities must add up to one.]

Outcomes	x	$f(x)$		$x \cdot f(x)$
R	**0**	$3/5$	= **0.6**	0.0
G R	**1**	$2/5 \cdot 3/4 = 6/20$	= **0.3**	0.3
G G R	**2**	$2/5 \cdot 1/4 \cdot 3/3 = 6/60$	= **0.1**	0.2
			1.0	0.5
				for part (b)

b) Suppose it costs \$5 to play the game, and the person gets \$10 for each green marble drawn. Is the game fair*? **Explain**. [Hint: Find E(X), the average (expected) number of green marbles drawn per game.]

$$\mu_X = E(X) = \sum_{\text{all } x} x \cdot f(x) = \mathbf{0.5}.$$

That is, **on average**, you get **0.5** green marbles per game. At \$10 per green marble, you win \$5 per game, **on average**. Since it costs \$5 to play the game, **on average**, you break even. **The game is fair**.

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* In gambling or betting, a game or situation in which the expected value of the profit for the player is zero (no net gain nor loss) is commonly called a "fair game."

OR

Let Y denote the net gain from playing 1 game. Then $\mathbf{Y = 10 \cdot X - 5}$.

$\mu_Y = E(Y) = 10 \cdot E(X) - 5 = 10 \cdot 0.5 - 5 = \$\mathbf{0}$.

The game is fair since the expected net gain is $**0**.
(i.e. on average, in the long run, you do not gain or do not lose money).

OR

Let Y denote the net gain from playing 1 game. Then $\mathbf{Y = 10 \cdot X - 5}$.

x	y	$f(x) = f(y)$	$y \cdot f(y)$
0	-5	0.6	-3.0
1	5	0.3	1.5
2	15	0.1	1.5
		1.0	**0.0**

$$\mu_Y = E(Y) = \sum_{\text{all } y} y \cdot f(y) = \$\mathbf{0}.$$

The game is fair since the expected net gain is $**0**.

(i.e. on average, in the long run, you do not gain or do not lose money).