Discussion 05 Solutions

1. **2.3-11** **2.5-3**

If the moment-generating function of X is

$$M_X(t) = \frac{2}{5}e^{t} + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t},$$

Find the mean, variance, and pmf of X.

$f(1) = \frac{2}{5}$, $f(2) = \frac{1}{5}$, $f(3) = \frac{2}{5}$.

$M_X'(t) = \frac{2}{5}e^{t} + \frac{2}{5}e^{2t} + \frac{6}{5}e^{3t}$. $E(X) = M_X'(0) = \mathbf{2}$.

OR

$E(X) = (1)\,\frac{2}{5} + (2)\,\frac{1}{5} + (3)\,\frac{2}{5} = \mathbf{2}$.

$M_X''(t) = \frac{2}{5}e^{t} + \frac{4}{5}e^{2t} + \frac{18}{5}e^{3t}$. $E(X^2) = M_X''(0) = \frac{24}{5} = 4.8$.

OR

$E(X^2) = (1)^2\,\frac{2}{5} + (2)^2\,\frac{1}{5} + (3)^2\,\frac{2}{5} = \frac{24}{5} = 4.8$.

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 4.8 - 2^2 = \mathbf{0.8}$.

2. Suppose a discrete random variable X has the following probability distribution:

$$f(0) = \frac{7}{8}, \qquad f(k) = \frac{1}{3^k}, \quad k = 2, 4, 6, 8, \ldots .$$

(possible values of X are even non-negative integers: $0, 2, 4, 6, 8, \ldots$).

Recall Discussion #2 Problem 1 (a): this is a valid probability distribution.

a) Find the moment-generating function of X, $M_X(t)$. For which values of t does it exist?

$$M_X(t) = E(e^{tX}) = e^{0t} \cdot \frac{7}{8} + \sum_{k=1}^{\infty} e^{2kt} \cdot \frac{1}{3^{2k}} = \frac{7}{8} + \sum_{k=1}^{\infty} \left(\frac{e^{2t}}{9} \right)^k$$

$$= \frac{7}{8} + \frac{\frac{e^{2t}}{9}}{1 - \frac{e^{2t}}{9}} = \frac{7}{8} + \frac{e^{2t}}{9 - e^{2t}} = \frac{9}{9 - e^{2t}} - \frac{1}{8}.$$

Must have $\left(\frac{e^{2t}}{9} \right) < 1$ for geometric series to converge. $\Rightarrow$ $t < \ln 3$.

b) Use $M_X(t)$ to find $E(X)$.

Recall Week 04 Discussion Problem 1 (a): $E(X) = \frac{9}{32}$.

$$M'_X(t) = \frac{2e^{2t}\left(9 - e^{2t}\right) - e^{2t}\left(-2e^{2t}\right)}{\left(9 - e^{2t}\right)^2} = \frac{18e^{2t}}{\left(9 - e^{2t}\right)^2}, \qquad t < \ln 3.$$

$$E(X) = M'_X(0) = \frac{18}{64} = \mathbf{\frac{9}{32}}.$$

3. Suppose a discrete random variable X has the following probability distribution:

$$f(1) = \ln 3 - 1, \qquad f(k) = \frac{(\ln 3)^k}{k!}, \quad k = 2, 3, 4, \ldots .$$

(possible values of X are positive integers: 1, 2, 3, 4, …).

Recall Discussion #2 Problem 1 (b): this is a valid probability distribution.

"Hint": Recall that $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$.

a) Find the moment-generating function of X, $M_X(t)$. For which values of t does it exist?

$$M_X(t) = \sum_{\text{all } x} e^{tx} \cdot p(x) = e^t \cdot (\ln 3 - 1) + \sum_{k=2}^{\infty} e^{tk} \cdot \frac{(\ln 3)^k}{k!}$$

$$= e^t \cdot (\ln 3 - 1) + \sum_{k=2}^{\infty} \frac{\left(e^t \ln 3\right)^k}{k!} = e^t \ln 3 - e^t + e^{e^t \ln 3} - 1 - e^t \ln 3$$

$$= 3^{e^t} - e^t - 1, \qquad t \in \mathbf{R}.$$

b) Use $M_X(t)$ to find $E(X)$.

Recall Week 04 Discussion Problem 1 (b): $E(X) = 3 \ln 3 - 1 \approx 2.2958$.

$$M_X'(t) = 3^{e^t} \cdot \ln 3 \cdot e^t - e^t, \qquad E(X) = M_X'(0) = \mathbf{3 \ln 3 - 1}.$$

4. Suppose the moment-generating function of X is

$$M_X(t) = 0.1\,e^{2t} + 0.3\,e^{4t} + 0.6\,e^{7t}.$$

a) Find $\mu = E(X)$.

$$M_X'(t) = 0.2\,e^{2t} + 1.2\,e^{4t} + 4.2\,e^{7t}.$$

$$E(X) = M_X'(0) = 0.2 + 1.2 + 4.2 = \mathbf{5.6}.$$

b) Find $\sigma = SD(X)$.

$$M_X''(t) = 0.4\,e^{2t} + 4.8\,e^{4t} + 29.4\,e^{7t}.$$

$$E(X^2) = M_X''(0) = 0.4 + 4.8 + 29.4 = 34.6.$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 34.6 - 5.6^2 = 3.24.$$

$$SD(X) = \sqrt{3.24} = \mathbf{1.8}.$$

OR

$$M_X(t) = 0.1\,e^{2t} + 0.3\,e^{4t} + 0.6\,e^{7t}. \quad \Rightarrow$$

x	$f(x)$
2	0.1
4	0.3
7	0.6

x	$f(x)$	$x \cdot f(x)$	$x^2 \cdot f(x)$	$(x-\mu)^2 \cdot f(x)$
2	0.1	0.2	0.4	1.296
4	0.3	1.2	4.8	0.768
7	0.6	4.2	29.4	1.176
		5.6	34.6	3.240

$\mu = E(X) = \sum x \cdot f(x) = \mathbf{5.6}.$ $\quad$ $\mathrm{Var}(X) = \sum (x-\mu)^2 \cdot f(x) = 3.24.$

OR $\quad$ $\mathrm{Var}(X) = \sum x^2 \cdot f(x) - \mu^2 = 34.6 - 5.6^2 = 3.24.$

$SD(X) = \sqrt{3.24} = \mathbf{1.8}.$

5. Suppose a discrete random variable X has the following probability distribution:

$$f(k) = \mathrm{P}(\mathrm{X} = k) = a^{k}, \quad k = 2, 3, 4, 5, 6, \ldots, \qquad \text{zero otherwise.}$$

a) Find the value of a that makes this is a valid probability distribution.

Must have $\sum_{\text{all } x} f(x) = 1$.

$$\Rightarrow \quad 1 = \sum_{k=2}^{\infty} a^{k} = \frac{\textit{first term}}{1 - \textit{base}} = \frac{a^2}{1-a}.$$

$$\Rightarrow \quad a^2 + a - 1 = 0. \qquad \Rightarrow \quad a = \frac{-1 \pm \sqrt{5}}{2}.$$

$$\frac{-1-\sqrt{5}}{2} < 0. \qquad \Rightarrow \quad a = \frac{\sqrt{5}-1}{2} \approx 0.618034.$$

Note: $a = \dfrac{1}{\varphi} = \varphi - 1$, where φ is the golden ratio.

b) Find $\mathrm{P}(\mathrm{X} \text{ is even})$.

$$\mathrm{P}(\mathrm{X} \text{ is even}) = f(2) + f(4) + f(6) + f(8) + \ldots$$

$$= a^2 + a^4 + a^6 + a^8 + \ldots = \frac{\textit{first term}}{1 - \textit{base}}$$

$$= \frac{a^2}{1-a^2} = \frac{a^2}{a} = a \approx 0.618034.$$

c) Find the moment-generating function of X, $M_X(t)$. For which values of t does it exist?

$$M_X(t) = E(e^{tX}) = \sum_{k=2}^{\infty} e^{tk} \cdot a^k = \sum_{k=2}^{\infty} \left(a \cdot e^t\right)^k = \frac{\textit{first term}}{1 - \textit{base}} = \frac{a^2 \cdot e^{2t}}{1 - a \cdot e^t},$$

$$\left(a \cdot e^t\right) < 1 \quad \Leftrightarrow \quad t < \ln \frac{1}{a} = \ln \varphi \approx 0.48121.$$

d) Find $E(X)$.

$$M_X'(t) = \frac{2a^2 \cdot e^{2t}\left(1 - a \cdot e^t\right) - a^2 \cdot e^{2t}\left(-a \cdot e^t\right)}{\left(1 - a \cdot e^t\right)^2} = \frac{2a^2 \cdot e^{2t} - a^3 \cdot e^{3t}}{\left(1 - a \cdot e^t\right)^2},$$

$$t < \ln \frac{1}{a}.$$

$$E(X) = M_X'(0) = \frac{2a^2 - a^3}{(1-a)^2} = \frac{2-a}{1-a} = 3 + a \approx 3.618034.$$

OR

$$E(X) = 2 \cdot a^2 + 3 \cdot a^3 + 4 \cdot a^4 + 5 \cdot a^5 + 6 \cdot a^6 + \ldots$$

$$a \cdot E(X) = \quad 2 \cdot a^3 + 3 \cdot a^4 + 4 \cdot a^5 + 5 \cdot a^6 + \ldots$$

$$\Rightarrow \quad (1-a) \cdot E(X) = a^2 + a^2 + a^3 + a^4 + a^5 + a^6 + \ldots = a^2 + 1.$$

Therefore, $E(X) = \dfrac{a^2 + 1}{1-a} = \dfrac{2-a}{1-a} = 1 + \dfrac{1}{1-a} = 3 + a \approx 3.618034.$

6. Let X be a continuous random variable with the probability density function

$$f(x) = \frac{C}{x^4}, \qquad x > 5, \qquad \text{zero otherwise.}$$

a) Find the value of C that would make $f(x)$ a valid probability density function.

Must have $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$1 = \int_5^{\infty} \frac{C}{x^4}\,dx = -\left.\frac{C}{3x^3}\right|_5^{\infty} = \frac{C}{375}. \qquad C = \mathbf{375}.$$

b) Find the cumulative distribution function of X, $F(x) = P(X \le x)$.

"Hint": Should be $F(5) = 0$, $F(\infty) = 1$.

$$F(x) = P(X \le x) = \int_5^{x} \frac{375}{u^4}\,du = -\left.\frac{375}{3u^3}\right|_5^{x} = 1 - \frac{125}{x^3}, \qquad x \ge 5.$$

c) Find the probability $P(6 < X < 10)$.

$$P(6 < X < 10) = \int_6^{10} \frac{375}{x^4}\,dx = -\left.\frac{375}{3x^3}\right|_6^{10} = \frac{125}{6^3} - \frac{125}{10^3}$$

$$\approx 0.5787 - 0.1250 = \mathbf{0.4537}.$$

OR

$$P(6 < X < 10) = F(10\text{-}) - F(6) = \left(1 - \frac{125}{10^3}\right) - \left(1 - \frac{125}{6^3}\right)$$

$$\approx 0.8750 - 0.4213 = \mathbf{0.4537}.$$

f) Find the 80th percentile of the distribution of X, $\pi_{0.80}$.

$$P(X \le \pi_{0.80}) = F(\pi_{0.80}) = 1 - \frac{125}{(\pi_{0.80})^3} = 0.80.$$

$$\pi_{0.80} = \sqrt[3]{625} \approx \mathbf{8.55}.$$

OR

$$P(X \ge \pi_{0.80}) = 0.20.$$

$$\int_{\pi_{0.80}}^{\infty} \frac{375}{x^4}\,dx = -\left.\frac{375}{3x^3}\right|_{\pi_{0.80}}^{\infty} = \frac{125}{(\pi_{0.80})^3} = 0.20.$$

$$\pi_{0.80} = \sqrt[3]{625} \approx \mathbf{8.55}.$$

For fun:

Median $= \sqrt[3]{250} \approx 6.3$.

g) Find the expected value of X, $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{5}^{\infty} x \cdot \frac{375}{x^4}\,dx = \int_{5}^{\infty} \frac{375}{x^3}\,dx = -\left.\frac{375}{2x^2}\right|_{5}^{\infty} = \mathbf{7.5}.$$

h) Find the standard deviation of X, $SD(X)$.

$$E(X) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_{5}^{\infty} x^2 \cdot \frac{375}{x^4}\,dx = \int_{5}^{\infty} \frac{375}{x^2}\,dx = -\left.\frac{375}{x}\right|_{5}^{\infty} = 75.$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 75 - 7.5^2 = 18.75.$$

$$SD(X) = \sqrt{18.75} \approx \mathbf{4.33}.$$

7. Let X be a continuous random variable with the probability density function

$$f(x) = C\,x^2, \qquad 3 \le x \le 9, \qquad \text{zero otherwise.}$$

a) Find the value of C that would make $f(x)$ a valid probability density function.

$$1 = \int_3^9 C\,x^2\,dx = \left.\frac{C\,x^3}{3}\right|_3^9 = \frac{729-27}{3}\,C = 234\,C. \qquad \Rightarrow \qquad \mathrm{C} = \mathbf{\frac{1}{234}}.$$

b) Find the probability $P(X < 5)$.

$$P(X<5) = \int_3^5 \frac{x^2}{234}\,dx = \left.\frac{x^3}{702}\right|_3^5 = \frac{125-27}{702} = \frac{98}{702} = \mathbf{\frac{49}{351}} \approx 0.1396.$$

c) Find the probability $P(X > 7)$.

$$P(X>7) = \int_7^9 \frac{x^2}{234}\,dx = \left.\frac{x^3}{702}\right|_7^9 = \frac{729-343}{702} = \frac{386}{702} = \mathbf{\frac{193}{351}} \approx 0.5499.$$

d) Find the mean of the probability distribution of X.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{3}^{9} x \cdot \frac{x^2}{234}\, dx = \left. \frac{x^4}{936} \right|_{3}^{9} = \frac{6480}{936} = \mathbf{\frac{90}{13}} \approx 6.923.$$

e) Find the median of the probability distribution of X.

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\, dy = \int_{3}^{x} \frac{y^2}{234}\, dy = \left. \frac{y^3}{702} \right|_{3}^{x} = \frac{x^3 - 27}{702}, \qquad 3 \le x < 9.$$

$$F(x) = P(X \le x) = 0, \qquad x < 3.$$

$$F(x) = P(X \le x) = 1, \qquad x \ge 9.$$

$$F(m) = \frac{1}{2}. \qquad \frac{m^3 - 27}{702} = \frac{1}{2}.$$

$$m^3 = \frac{702}{2} + 27 = 378. \qquad m = \mathbf{\sqrt[3]{378}} \approx 7.23.$$

8. Suppose a random variable X has the following probability density function:

$$f(x) = \cos x, \qquad 0 < x < \frac{\pi}{2}, \qquad \text{zero otherwise.}$$

a) Find $P(X < \frac{\pi}{4})$.

$$P(X < \frac{\pi}{4}) = \int_0^{\pi/4} \cos x \, dx = (\sin x)\Big|_0^{\pi/4} = \sin\frac{\pi}{4} - 0 = \mathbf{\frac{\sqrt{2}}{2}} \approx 0.7071.$$

b) Find $\mu = E(X)$.

$$\mu = E(X) = \int_0^{\pi/2} x \cdot \cos x \, dx = (x \cdot \sin x + \cos x)\Big|_0^{\pi/2} = \mathbf{\frac{\pi}{2} - 1} \approx 0.5708.$$

c) Find the median of the probability distribution of X.

$$F(x) = P(X \le x) = \int_0^{x} \cos y \, dy = \sin x, \qquad 0 < x < \frac{\pi}{2}.$$

Median: $\quad F(m) = P(X \le m) = \frac{1}{2}.$

$$\Rightarrow \quad \sin m = \frac{1}{2}. \qquad m = \mathbf{\frac{\pi}{6}} \approx 0.5236.$$

9. Let X be a continuous random variable with the probability density function

$$f(x) = 6x(1-x), \qquad 0 < x < 1, \qquad \text{zero elsewhere.}$$

Compute $P(\mu - 2\sigma < X < \mu + 2\sigma)$.

$$\mu = E(X) = \int_0^1 x \cdot 6x(1-x)\,dx = \int_0^1 6x^2\,dx - \int_0^1 6x^3\,dx = \frac{6}{3} - \frac{6}{4} = \frac{1}{2} = 0.50.$$

$$E(X^2) = \int_0^1 x^2 \cdot 6x(1-x)\,dx = \int_0^1 6x^3\,dx - \int_0^1 6x^4\,dx = \frac{6}{4} - \frac{6}{5} = 0.30.$$

$$\sigma^2 = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 0.30 - 0.25 = 0.05.$$

$$\sigma = \sqrt{0.05} \approx 0.2236.$$

$$\mu - 2\sigma \approx 0.0528, \qquad \mu + 2\sigma \approx 0.9472.$$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx \int_{0.0528}^{0.9472} 6x(1-x)\,dx = \left(3x^2 - 2x^3\right)\Big|_{0.0528}^{0.9472}$$

$$\approx 0.992 - 0.008 = \mathbf{0.984}.$$

10. Suppose a random variable X has the following probability density function:

$$f(x) = x\,e^{x}, \qquad 0 < x < 1, \qquad \text{zero otherwise.}$$

a) Find $P(X < \frac{1}{2})$.

$$P(X < \tfrac{1}{2}) = \int_0^{1/2} x\,e^{x}\,dx = \left[x\,e^{x} - e^{x} \right]\Big|_0^{1/2} = 1 - \frac{\sqrt{e}}{2} \approx 0.175639.$$

b) Find $\mu = E(X)$.

$$\mu = E(X) = \int_0^{1} x \cdot x\,e^{x}\,dx = \left[x^2 e^{x} - 2x\,e^{x} + 2e^{x} \right]\Big|_0^{1} = \boldsymbol{e - 2}.$$

c) Find the moment-generating function of X, $M_X(t)$.

$$M_X(t) = \int_0^{1} e^{tx} \cdot x\,e^{x}\,dx = \int_0^{1} x\,e^{(t+1)x}\,dx$$

$$= \left[\frac{1}{t+1} x\,e^{(t+1)x} - \frac{1}{(t+1)^2} e^{(t+1)x} \right]\Big|_0^{1}$$

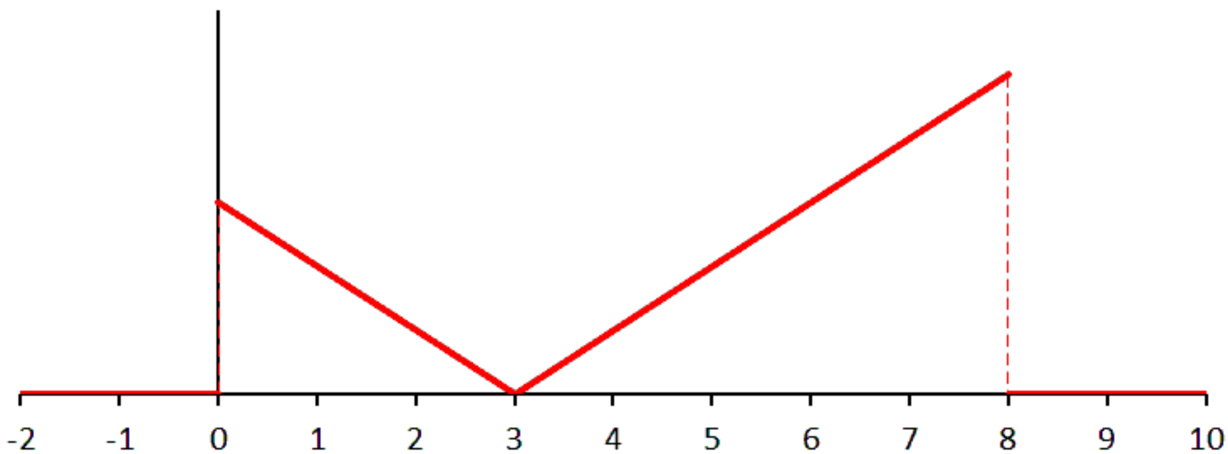
$$= \frac{1}{t+1} e^{t+1} - \frac{1}{(t+1)^2} e^{t+1} + \frac{1}{(t+1)^2}$$

$$= \frac{t\,e^{t+1} + 1}{(t+1)^2}, \qquad t \neq -1.$$

$$M_X(-1) = \int_0^{1} x\,dx = \frac{1}{2}.$$

11. Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} c\,|\,x-3\,|, & 0 < x < 8, \\ 0, & \text{otherwise.} \end{cases}$$



a) Find the value of c that makes $f(x)$ a valid probability density function.

Must have $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$1 = \int_0^8 c\,|\,x-3\,|\,dx = c\int_0^3 (3-x)\,dx + c\int_3^8 (x-3)\,dx = \frac{9}{2}c + \frac{25}{2}c = 17c.$$

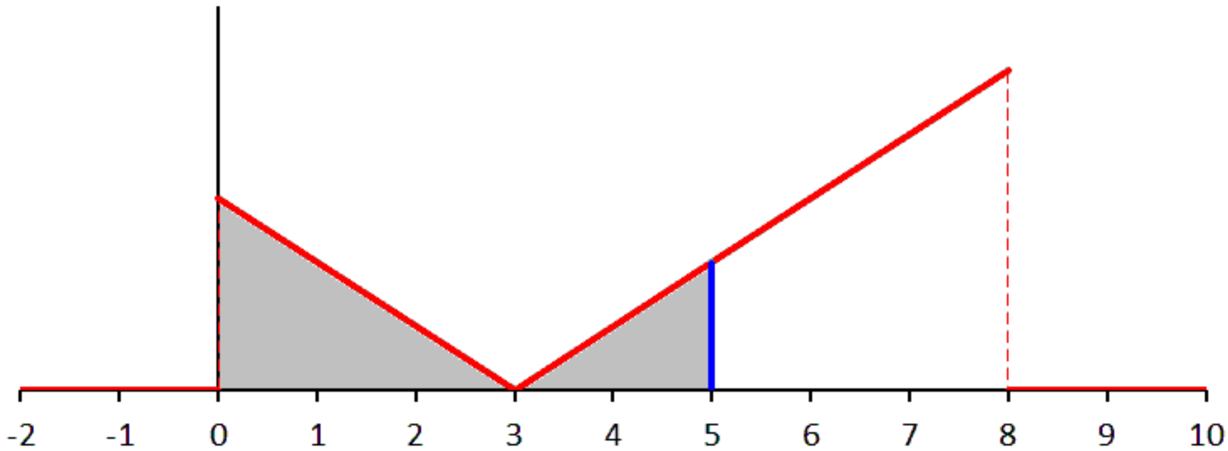
$$\Rightarrow \quad 1 = 17c. \qquad \Rightarrow \quad c = \mathbf{\frac{1}{17}}.$$

b) Find the probability $P(X < 5)$.

$$P(X<5) = \int_0^5 \frac{1}{17}|\,x-3\,|\,dx = \frac{1}{17}\int_0^3 (3-x)\,dx + \frac{1}{17}\int_3^5 (x-3)\,dx = \frac{9}{34} + \frac{4}{34} = \mathbf{\frac{13}{34}}.$$

OR

$$P(X<5) = 1 - P(X \ge 5) = 1 - \int_5^8 \frac{1}{17}|\,x-3\,|\,dx = 1 - \frac{1}{17}\int_5^8 (x-3)\,dx = 1 - \frac{21}{34} = \mathbf{\frac{13}{34}}.$$



c) Find the median of the probability distribution of X.

$$F_X(x) = 0, \qquad x < 0,$$

$$F_X(x) = \frac{1}{17}\int_0^x (3-y)\,dy = \frac{9}{34} - \frac{(3-x)^2}{34}, \qquad 0 \le x < 3,$$

$$F_X(x) = \frac{1}{17}\int_0^3 (3-y)\,dy + \frac{1}{17}\int_3^x (y-3)\,dy = \frac{9}{34} + \frac{(x-3)^2}{34}, \qquad 3 \le x < 8,$$

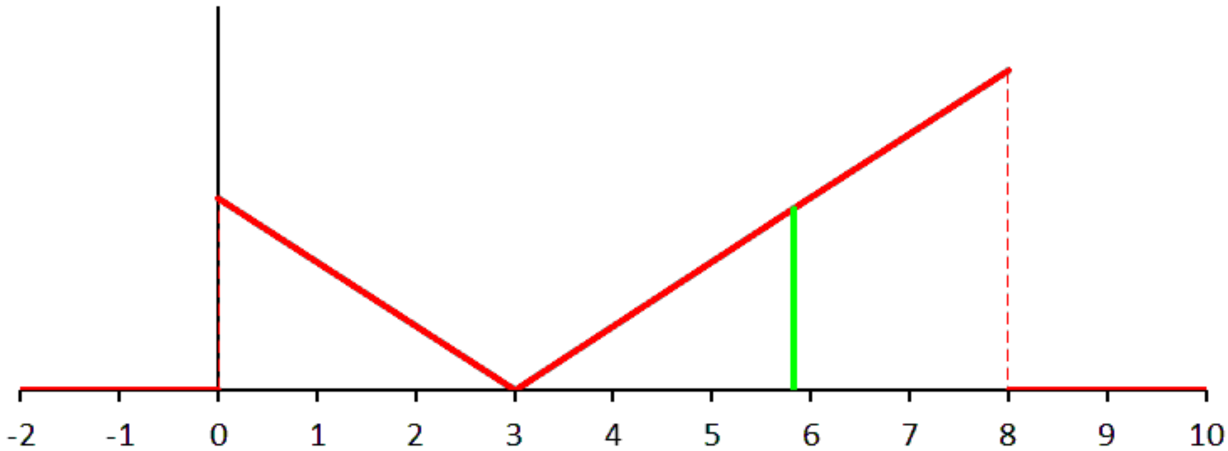
$$F_X(x) = 1, \qquad x \ge 8.$$

$$F_X(3) = \frac{9}{34}. \qquad \Rightarrow \qquad 3 < \text{median} < 8.$$

$$F_X(\text{median}) = \frac{9}{34} + \frac{(\text{median}-3)^2}{34} = \frac{1}{2}.$$

$$\Rightarrow \qquad (\text{median}-3)^2 = 8.$$

$$\Rightarrow \qquad \text{median} = \mathbf{3+\sqrt{8}} \approx 5.828427.$$



d) Find the mean of the probability distribution of X.

$$\mathrm{E}(\mathrm{X}) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \frac{1}{17}\int_0^3 x(3-x)\,dx + \frac{1}{17}\int_3^8 x(x-3)\,dx = \frac{9}{34} + \frac{475}{102}$$

$$= \frac{502}{102} = \mathbf{\frac{251}{51}} \approx 4.92157.$$

e) Find the variance of the probability distribution of X.

$$\mathrm{E}(\mathrm{X}^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \frac{1}{17}\int_0^3 x^2(3-x)\,dx + \frac{1}{17}\int_3^8 x^2(x-3)\,dx = \frac{27}{68} + \frac{2075}{68}$$

$$= \frac{1051}{34} \approx 30.912.$$

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(\mathrm{X}^2) - [\mathrm{E}(\mathrm{X})]^2 = \frac{1051}{34} - \left(\frac{251}{51}\right)^2 = \mathbf{\frac{34801}{5202}} \approx 6.69.$$