STAT 400 Spring 2018

Discussion 07

1. Suppose that number of accidents at a construction site follows a Poisson process with the average rate of 0.80 accidents per month. Assume all months are independent of each other.

a) Find the probability that the first accident of a calendar year would occur during March.

b) Find the probability that the third accident of a calendar year would occur during April.

c) Find the probability that the third accident of a calendar year would occur during spring (March, April, or May).

"Hint": If T_α has a Gamma$(\alpha, \theta = 1/\lambda)$ distribution, where α is an integer, then
$F_{T_\alpha}(t) = P(T_\alpha \le t) = P(X_t \ge \alpha)$ and $P(T_\alpha > t) = P(X_t \le \alpha - 1)$,
where X_t has a Poisson(λt) distribution.

2. As Alex is leaving for college, his parents give him a car, but warn him that they would take the car away if Alex gets 6 speeding tickets. Suppose that Alex receives speeding tickets according to Poisson process with the average rate of one ticket per six months.

a) Find the probability that it would take Alex longer than two years to get his sixth speeding ticket.

b) Find the probability that it would take Alex less than four years to get his sixth speeding ticket.

c) Find the probability that Alex would get his sixth speeding ticket during the fourth year.

d) Find the probability that Alex would get his sixth speeding ticket during the third year.

3. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x,y) = C(x+2y), \quad 0<x<2, \; 0<y<3, \quad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0<x<2, \; 0<y<3 \}$.

b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

4. Let X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = Cx^2y, \quad 0<x<4, \; 0<y<\sqrt{x}, \quad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ 0<x<4, \; 0<y<\sqrt{x} \}$.

b) Find the value of C so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

e)* Are X and Y independent?

If X and Y are not independent, find $\mathrm{Cov}(X, Y)$.

5. Let the joint probability density function for (X, Y) be

$$f(x,y) = x+y, \quad x>0, \quad y>0, \quad x+2y<2, \quad \text{zero otherwise.}$$

a) Find the probability $P(Y > X)$.

b) Find the marginal p.d.f. of X, $f_X(x)$.

c) Find the marginal p.d.f. of Y, $f_Y(y)$.

d)* Are X and Y independent? If not, find $\mathrm{Cov}(X, Y)$.

6 – 9. Let the joint probability density function for (X, Y) be

$$f(x,y) = \frac{12}{5} x y^3, \qquad 0 < y < 1, \quad y < x < 2, \qquad \text{zero otherwise.}$$

Do NOT use a computer. You may only use +, –, ×, ÷, and $\sqrt{\ }$ on a calculator. Show all work. Example:

$$\int_0^1 \left(\int_y^2 \frac{12}{5} x y^3 \, dx \right) dy = \int_0^1 \left(\frac{6}{5} x^2 y^3 \right) \Bigg|_{x=y}^{x=2} dy = \int_0^1 \left(\frac{24}{5} y^3 - \frac{6}{5} y^5 \right) dy$$

$$= \left(\frac{6}{5} y^4 - \frac{1}{5} y^6 \right) \Bigg|_{y=0}^{y=1} = \frac{6}{5} - \frac{1}{5} = 1. \qquad \Rightarrow \qquad f(x,y) \text{ is a valid joint p.d.f.}$$

6. a) Sketch the support of (X, Y). That is, sketch $\{ 0 < y < 1, \ y < x < 2 \}$.

b) Find the marginal probability density function of X, $f_X(x)$.

c) Find the marginal probability density function of Y, $f_Y(y)$.

d) Are X and Y independent? Justify your answer.

7. Find the probability $P(X > 2Y)$.

a) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

b) Set up the double integral(s) over the region that "we want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

c) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. x and the inside integral w.r.t. y.

d) Set up the double integral(s) over the region that "we do not want" with the outside integral w.r.t. y and the inside integral w.r.t. x.

e) Use one of (a) – (d) to find the desired probability.

8. Find the probability $P(X + Y < 2)$. Repeat parts (a) – (e) of problem 7.

9. Find the probability $P(XY < 1)$. Repeat parts (a) – (e) of problem 7.

10. Let the joint probability density function for (X, Y) be

$$f(x, y) = C\,x\,y, \qquad x > 0, \quad y > 0, \quad x^2 + (y+3)^2 < 25,$$
$$\text{zero elsewhere.}$$

a) Find the value of C so that $f(x, y)$ is a valid joint p.d.f.

b) Find $P(2X + Y > 2)$.

c) Find $P(X - 3Y > 0)$.

11. Suppose that (X, Y) is uniformly distributed over the region defined by $x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 1$. That is,

$$f(x, y) = C, \qquad x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 1, \qquad \text{zero elsewhere.}$$

a) What is the joint probability density function of X and Y? That is, find the value of C so that $f(x, y)$ is a valid joint p.d.f.

b) Find $P(X + Y < 1)$.

c) Find $P(Y > 2X)$.

d)* Are X and Y independent?

12. Consider two continuous random variables X and Y with joint p.d.f.

$$f_{X,Y}(x, y) = \frac{C}{(2x + y)^3}, \qquad y > 1,\ 0 < x < y, \qquad \text{zero elsewhere.}$$

a) Sketch the support of (X, Y). That is, sketch $\{ y > 1,\ 0 < x < y \}$.

b) Find the value of C so that $f_{X,Y}(x, y)$ is a valid joint p.d.f.

c) Find the marginal probability density function of X, $f_X(x)$.

d) Find the marginal probability density function of Y, $f_Y(y)$.

e) Find $P(X + Y < 2)$.

f) Find $P(X + Y > 5)$.

g) Find $P(Y > 3X)$.

h)* Are X and Y independent?