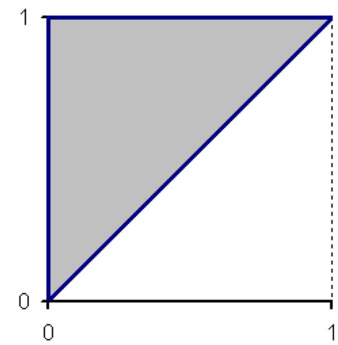
Discussion 08 Solutions

1. Let X and Y have the joint probability density function

$$f_{X,Y}(x,y) = C\,x\,y^2, \qquad 0 < x < y < 1,$$

zero otherwise.



a) What must the value of C be so that $f_{X,Y}(x,y)$ is a valid joint p.d.f.?

$$1 = \int_0^1 \left(\int_0^y C\,x\,y^2\,dx \right) dy = \int_0^1 \frac{C}{2} y^4\,dy = \frac{C}{10}. \qquad \Rightarrow \qquad C = \mathbf{10}.$$

OR

$$1 = \int_0^1 \left(\int_x^1 C\,x\,y^2\,dy \right) dx = \int_0^1 \frac{C}{3}\left(x - x^4\right) dx = \frac{C}{3}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{C}{10}.$$

$$\Rightarrow \qquad C = \mathbf{10}.$$

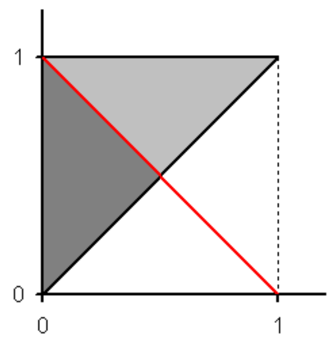
b) Find the marginal probability density function of X, $f_X(x)$. *Include its support.*

$$f_X(x) = \int_x^1 10\,x\,y^2\,dy = \frac{10}{3} x\left(1 - x^3\right) = \boldsymbol{\frac{10}{3}\left(x - x^4\right)}, \qquad \boldsymbol{0 < x < 1}.$$

c) Find the marginal probability density function of Y, $f_Y(y)$. *Include its support.*

$$f_Y(y) = \int_0^y 10\,x\,y^2\,dx = \boldsymbol{5\,y^4}, \qquad \boldsymbol{0 < y < 1}.$$

d) Find $P(X+Y<1)$.



$$P(X+Y<1) = \int_0^{1/2}\left(\int_x^{1-x} 10\,x\,y^2\,dy\right)dx$$

$$= \int_0^{1/2}\left(\frac{10}{3}x\int_x^{1-x} 3\,y^2\,dy\right)dx$$

$$= \int_0^{1/2}\frac{10}{3}x\left((1-x)^3 - x^3\right)dx$$

$$= \int_0^{1/2}\frac{10}{3}x\left(1-3x+3x^2-2x^3\right)dx = \int_0^{1/2}\left(\frac{10}{3}x - 10x^2 + 10x^3 + \frac{20}{3}x^4\right)dx$$

$$= \left(\frac{5}{3}x^2 - \frac{10}{3}x^3 + \frac{5}{2}x^4 - \frac{4}{3}x^5\right)\Bigg|_0^{1/2} = \left(\frac{5}{12} - \frac{10}{24} + \frac{5}{32} - \frac{4}{96}\right) = \mathbf{\frac{11}{96}}.$$

OR

$$P(X+Y<1) = 1 - \int_{1/2}^{1}\left(\int_{1-y}^{y} 10\,x\,y^2\,dx\right)dy = 1 - \int_{1/2}^{1} 5\,y^2\left(\int_{1-y}^{y} 2\,x\,dx\right)dy$$

$$= 1 - \int_{1/2}^{1} 5\,y^2\left(y^2 - (1-y)^2\right)dy = 1 - \int_{1/2}^{1} 5\,y^2\,(2\,y-1)\,dy$$

$$= 1 - \int_{1/2}^{1}\left(10\,y^3 - 5\,y^2\right)dy = 1 - \left(\frac{5}{2}y^4 - \frac{5}{3}y^3\right)\Bigg|_{1/2}^{1}$$

$$= 1 - \left(\frac{5}{2} - \frac{5}{3}\right) + \left(\frac{5}{32} - \frac{5}{24}\right) = \mathbf{\frac{11}{96}}.$$

OR

$$P(X+Y<1) = \int_0^{1/2}\left(\int_0^{y} 10\,x\,y^2\,dx\right)dy + \int_{1/2}^{1}\left(\int_0^{1-y} 10\,x\,y^2\,dx\right)dy$$

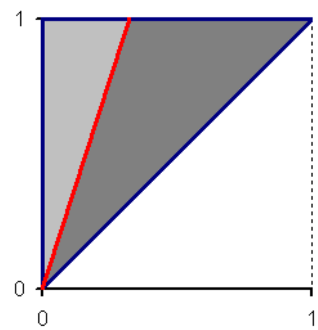
$$= \int_0^{1/2} 5\,y^4\,dy + \int_{1/2}^{1} 5\,y^2\,(1-y)^2\,dy = \frac{1}{32} + \int_{1/2}^{1}\left(5\,y^2 - 10\,y^3 + 5\,y^4\right)dy$$

$$= \frac{1}{32} + \left(\frac{5}{3}y^3 - \frac{5}{2}y^4 + y^5\right)\Bigg|_{1/2}^{1} = \frac{1}{32} + \frac{1}{6} - \left(\frac{5}{24} - \frac{5}{32} + \frac{1}{32}\right) = \mathbf{\frac{11}{96}}.$$

e) Let $a>1$. Find $P(Y<a\,X)$.

$$1 - \int_0^{1}\left(\int_0^{y/a} 10\,x\,y^2\,dx\right)dy = 1 - \int_0^{1} 5\,\frac{y^4}{a^2}\,dy$$

$$= 1 - \frac{1}{a^2}.$$



OR $$1 - \int_0^{1/a}\left(\int_{a\,x}^{1} 10\,x\,y^2\,dy\right)dx = \ldots = 1 - \frac{1}{a^2}.$$

OR $$\int_0^{1}\left(\int_{y/a}^{y} 10\,x\,y^2\,dx\right)dy = \ldots = 1 - \frac{1}{a^2}.$$

OR $$\int_0^{1/a}\left(\int_x^{a\,x} 10\,x\,y^2\,dy\right)dx + \int_{1/a}^{1}\left(\int_x^{1} 10\,x\,y^2\,dy\right)dx = \ldots = 1 - \frac{1}{a^2}.$$

f) Are X and Y independent? If not, find $\text{Cov}(X, Y)$.

$$f_{X,Y}(x, y) \neq f_X(x) \cdot f_Y(y). \quad \Rightarrow \quad \text{X and Y are } \textbf{NOT independent}.$$

OR

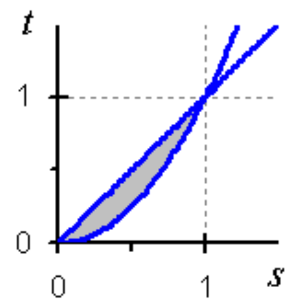
The support of (X, Y) is NOT a rectangle. $\Rightarrow$ X and Y are **NOT independent**.

$$E(X) = \frac{5}{9}, \qquad E(Y) = \frac{5}{6}, \qquad E(XY) = \frac{10}{21},$$

$$\text{Cov}(S, T) = \frac{10}{21} - \frac{5}{9} \cdot \frac{5}{6} = \boldsymbol{\frac{5}{378}} \approx 0.0132275.$$

2. Let S and T have the joint probability density function

$$f_{S,T}(s, t) = \frac{1}{t}, \qquad 0 < s < 1, \quad s^2 < t < s.$$



a) Find $f_S(s)$ and $f_T(t)$.

$$f_S(s) = \int_{s^2}^{s} \frac{1}{t}\, dt = \left(\ln t\right)\Big|_{s^2}^{s} = \ln s - \ln s^2 = -\ln s, \qquad 0 < s < 1.$$

$$f_T(t) = \int_{t}^{\sqrt{t}} \frac{1}{t}\, ds = \frac{1}{t}\left(\sqrt{t} - t\right) = \frac{1}{\sqrt{t}} - 1, \qquad 0 < t < 1.$$

b) Find $E(S)$ and $E(T)$.

$$E(S) = \int_0^1 s(-\ln s)\,ds = \left(-\frac{s^2}{2}\ln s + \frac{s^2}{4}\right)\Bigg|_0^1 = \mathbf{\frac{1}{4}}.$$

$$E(T) = \int_0^1 t\left(\frac{1}{\sqrt{t}} - 1\right)dt = \int_0^1 \left(\sqrt{t} - t\right)dt = \frac{2}{3} - \frac{1}{2} = \mathbf{\frac{1}{6}}.$$

c) Find the correlation coefficient ρ_{ST}.

$$E(ST) = \int_0^1 \left(\int_{s^2}^{s} s\,t\,\frac{1}{t}\,dt\right)ds = \int_0^1 s\left(\int_{s^2}^{s} dt\right)ds = \int_0^1 \left(s^2 - s^3\right)ds = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

$$\text{Cov}(S,T) = \frac{1}{12} - \frac{1}{4}\cdot\frac{1}{6} = \frac{1}{24}.$$

$$E(S^2) = \int_0^1 s^2(-\ln s)\,ds = \left(-\frac{s^3}{3}\ln s + \frac{s^3}{9}\right)\Bigg|_0^1 = \frac{1}{9}.$$

$$\text{Var}(S) = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}.$$

$$E(T^2) = \int_0^1 t^2\left(\frac{1}{\sqrt{t}} - 1\right)dt = \int_0^1 \left(t^{3/2} - t^2\right)dt = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}.$$

$$\text{Var}(T) = \frac{1}{15} - \frac{1}{36} = \frac{7}{180}.$$

$$\rho_{ST} = \frac{1/24}{\sqrt{7/144}\times\sqrt{7/180}} = \frac{3\sqrt{5}}{7} \approx \mathbf{0.9583}.$$

3. Let X and Y be random variables with

$$E(X) = \mu_X = 25, \quad SD(X) = \sigma_X = 4,$$

$$E(Y) = \mu_Y = 40, \quad SD(Y) = \sigma_Y = 3, \qquad \mathrm{Corr}(X, Y) = \rho = -0.50.$$

a) Find $E(2X + 5Y)$ and $SD(2X + 5Y)$.

$$E(2X + 5Y) = 2\mu_X + 5\mu_Y = 2 \cdot 25 + 5 \cdot 40 = \mathbf{250}.$$

$$\begin{aligned}
\mathrm{Var}(2X + 5Y) &= \mathrm{Cov}(2X + 5Y, 2X + 5Y) \\
&= \mathrm{Cov}(2X, 2X) + \mathrm{Cov}(2X, 5Y) + \mathrm{Cov}(5Y, 2X) + \mathrm{Cov}(5Y, 5Y) \\
&= 4\sigma_X^2 + 20\sigma_{XY} + 25\sigma_Y^2 = 4\sigma_X^2 + 20\rho\sigma_X\sigma_Y + 25\sigma_Y^2 \\
&= 4 \cdot 4^2 + 20 \cdot (-0.50) \cdot 4 \cdot 3 + 25 \cdot 3^2 = 169.
\end{aligned}$$

$$SD(2X + 5Y) = \sqrt{169} = \mathbf{13}.$$

b) Find $E(4Y - 5X)$ and $SD(4Y - 5X)$.

$$E(4Y - 5X) = 4\mu_Y - 5\mu_X = 4 \cdot 40 - 5 \cdot 25 = \mathbf{35}.$$

$$\begin{aligned}
\mathrm{Var}(4Y - 5X) &= \mathrm{Cov}(4Y - 5X, 4Y - 5X) \\
&= \mathrm{Cov}(4Y, 4Y) - \mathrm{Cov}(4Y, 5X) - \mathrm{Cov}(5X, 4Y) + \mathrm{Cov}(5X, 5X) \\
&= 16\sigma_Y^2 - 40\sigma_{XY} + 25\sigma_X^2 = 16\sigma_Y^2 - 40\rho\sigma_X\sigma_Y + 25\sigma_X^2 \\
&= 16 \cdot 3^2 - 40 \cdot (-0.50) \cdot 4 \cdot 3 + 25 \cdot 4^2 = 784.
\end{aligned}$$

$$SD(4Y - 5X) = \sqrt{784} = \mathbf{28}.$$

4. One piece of PVC pipe is to be inserted inside another piece. The length of the first piece is normally distributed with mean value 25 in. and standard deviation 0.9 in. The length of the second piece is a normal random variable with mean and standard deviation 20 in. and 0.6 in., respectively. The amount of overlap is normally distributed with mean value 1 in. and standard deviation 0.2 in. Assuming that the lengths and amount of overlap are independent of one another, what is the probability that the total length after insertion is between 43.45 in. and 45.65 in.?

Total = First + Second – Overlap.

$$\text{E(Total)} = \text{E(First)} + \text{E(Second)} - \text{E(Overlap)} = 25 + 20 - 1 = 44.$$

$$\text{Var(Total)} = \text{Var(First)} + \text{Var(Second)} + (-1)^2\,\text{Var(Overlap)}$$

$$= 0.9^2 + 0.6^2 + 0.2^2 = 0.81 + 0.36 + 0.04 = 1.21.$$

$\text{SD(Total)} = \sqrt{1.21} = 1.1.$ Total has Normal distribution.

$$P(\,43.45 < \text{Total} < 45.65\,) = P\left(\frac{43.45-44}{1.1} < Z < \frac{45.65-44}{1.1}\right) = P(\,-0.50 < Z < 1.50\,)$$

$$= 0.9332 - 0.3085 = \mathbf{0.6247}.$$

5. A gas station sells three grades of gasoline: regular unleaded, extra unleaded, and super unleaded. These are priced at \$1.55, \$1.70, and \$1.85 per gallon *, respectively. Let X_1, X_2, and X_3 denote the amounts of these grades purchased (gallons) on a particular day. Suppose the X_i's are independent with $\mu_1 = 1{,}000$, $\mu_2 = 500$, $\mu_3 = 300$, $\sigma_1 = 100$, $\sigma_2 = 80$, and $\sigma_3 = 50$. If the X_i's are normally distributed, what is the probability that revenue exceeds …

a) \$2,600? b) \$3,000?

$\text{Total} = 1.55\,X_1 + 1.70\,X_2 + 1.85\,X_3.$

$$\begin{aligned} E(\text{Total}) &= 1.55\,E(X_1) + 1.70\,E(X_2) + 1.85\,E(X_3) \\ &= 1.55 \times 1{,}000 + 1.70 \times 500 + 1.85 \times 300 = \$2{,}955. \end{aligned}$$

$$\begin{aligned} \text{Var}(\text{Total}) &= (1.55)^2\,\text{Var}(X_1) + (1.70)^2\,\text{Var}(X_2) + (1.85)^2\,\text{Var}(X_3) \\ &= 1.55^2 \times 100^2 + 1.70^2 \times 80^2 + 1.85^2 \times 50^2 = 51{,}077.25. \end{aligned}$$

$\text{SD}(\text{Total}) = \sqrt{51{,}077.25} \approx \$226.$ Total has Normal distribution.

a) $P(\text{Total} > 2{,}600) = P\left(Z > \dfrac{2{,}600 - 2{,}955}{226}\right) = P(Z > -1.57) = 1 - 0.0582 = \mathbf{0.9418}.$

b) $P(\text{Total} > 3{,}000) = P\left(Z > \dfrac{3{,}000 - 2{,}955}{226}\right) = P(Z > 0.20) = 1 - 0.5793 = \mathbf{0.4207}.$

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* This problem was written long time ago.

6. Suppose that the actual weight of "10-pound" sacks of potatoes varies from sack to sack and that the actual weight may be considered a random variable having a normal distribution with the mean of 10.2 pounds and the standard deviation of 0.6 pounds. Similarly, the actual weight of "3-pound" bags of apples varies from bag to bag and that the actual weight may be considered a random variable having a normal distribution with the mean of 3.15 pounds and the standard deviation of 0.3 pounds. A boy-scout troop is planning a camping trip. If the boy-scouts buy 3 "10-pound" sacks of potatoes and 4 "3-pound" bags of apples selecting them at random, what is the probability that the total weight would exceed 42 pounds?

Since weights vary from sack to sack and from bag to bag,

$\text{Total} = P_1 + P_2 + P_3 + A_1 + A_2 + A_3 + A_4.$

$$E(\text{Total}) = E(P_1) + E(P_2) + E(P_3) + E(A_1) + E(A_2) + E(A_3) + E(A_4)$$

$$= 10.2 + 10.2 + 10.2 + 3.15 + 3.15 + 3.15 + 3.15 = 43.2.$$

$$\text{Var}(\text{Total}) = \text{Var}(P_1) + \text{Var}(P_2) + \text{Var}(P_3) + \text{Var}(A_1) + \text{Var}(A_2) + \text{Var}(A_3) + \text{Var}(A_4)$$

$$= 0.6^2 + 0.6^2 + 0.6^2 + 0.3^2 + 0.3^2 + 0.3^2 + 0.3^2 = 1.44.$$

$\text{SD}(\text{Total}) = \sqrt{1.44} = 1.2.$ Total has Normal distribution.

$$P(\text{ Total} > 42 \text{ }) = P\left(Z > \frac{42 - 43.2}{1.2} \right) = P(Z > -1.00) = \mathbf{0.8413}.$$

7. Every month, the government of Neverland spends X million dollars purchasing guns and Y million dollars purchasing butter. Assume X and Y are independent, X has a Normal distribution with mean 265 and standard deviation 40 (in millions of dollars), and Y has a Normal distribution with mean 170 and standard deviation 30 (in millions of dollars).

a) Find the probability that the government of Neverland spends more on guns than on butter during a given month. That is, find $P(X > Y)$.

$P(X > Y) = P(X - Y > 0)$.

$X - Y$ has Normal distribution with mean $E(X - Y) = 265 - 170 = 95$

and variance $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) = 40^2 + 30^2 = 2500$

(standard deviation $= 50$).

$$P(X > Y) = P(X - Y > 0) = P\left(Z > \frac{0 - 95}{50}\right) = P(Z > -1.90) = \mathbf{0.9713}.$$

b) Find the probability that the government of Neverland spends more on guns than twice the amount it spends on butter during a given month. That is, find $P(X > 2Y)$.

$P(X > 2Y) = P(X - 2Y > 0)$.

$X - 2Y$ has Normal distribution with mean $E(X - 2Y) = 265 - 2 \cdot 170 = -75$

and variance $\text{Var}(X - 2Y) = \text{Var}(X) + 4 \cdot \text{Var}(Y) = 40^2 + 4 \cdot 30^2 = 5200$

(standard deviation ≈ 72.111).

$$P(X > 2Y) = P(X - 2Y > 0) = P\left(Z > \frac{0 + 75}{72.111}\right) = P(Z > 1.04) = \mathbf{0.1492}.$$

c) Find the probability that the government of Neverland exceeds the 500-million spending limit during a given month. That is, find $P(X + Y > 500)$.

$X + Y$ has Normal distribution with mean $E(X + Y) = 265 + 170 = 435$

and variance $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = 40^2 + 30^2 = 2500$

(standard deviation $= 50$).

$$P(X+Y>500) = P\left(Z>\frac{500-435}{50}\right) = P(Z>1.30) = \mathbf{0.0968}.$$

8. The previous problem is not very realistic – X and Y should NOT be independent, but the correlation coefficient of X and Y should be negative. Assume X has a Normal distribution with mean 265 and standard deviation 40 (in millions of dollars), and Y has a Normal distribution with mean 170 and standard deviation 30 (in millions of dollars). Assume also that the correlation coefficient of X and Y is $\rho = -0.56$. Assume that any linear combination of X and Y is normally distributed (that would be the case if X and Y jointly have a Bivariate Normal distribution [**4.5** **4.4**]).

a) Find the probability that the government of Neverland spends more on guns than on butter during a given month. That is, find $P(X > Y)$.

$$E(X - Y) = 265 - 170 = 95.$$

$$\mathrm{Var}(X - Y) = \mathrm{Var}(X) - 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y) = \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2$$

$$= 40^2 - 2 \cdot (-0.56) \cdot 40 \cdot 30 + 30^2 = 3844$$

(standard deviation $= 62$).

$$P(X > Y) = P(X - Y > 0) = P\left(Z > \frac{0 - 95}{62}\right) = P(Z > -1.53) = \mathbf{0.9370}.$$

b) Find the probability that the government of Neverland exceeds the 500-million spending limit during a given month. That is, find $P(X + Y > 500)$.

$$E(X + Y) = 265 + 170 = 435.$$

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y) = \sigma_X^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y^2$$

$$= 40^2 + 2 \cdot (-0.56) \cdot 40 \cdot 30 + 30^2 = 1156$$

(standard deviation $= 34$).

$$P(X + Y > 500) = P\left(Z > \frac{500 - 435}{34}\right) = P(Z > 1.91) = \mathbf{0.0281}.$$

"Hint": In each case, find the mean and the variance of the appropriate linear combination of X and Y first.