STAT 400 **Discussion 13 Solutions** Spring 2018

1. In a random sample of 200 students who took Exam P in 2017, 74 successfully passed the exam. (Exam P [Probability] is the first Actuarial Science exam.)

a) Construct a 95% confidence interval for the overall proportion of students who successfully passed Exam P in 2017.

The confidence interval: $\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\dfrac{\hat{p} \cdot (1-\hat{p})}{n}}$. $\hat{p} = \dfrac{Y}{n} = \dfrac{74}{200} = 0.37.$

95% confidence level, $z_{\alpha/2} = z_{0.025} = 1.96.$

$0.37 \pm 1.96 \cdot \sqrt{\dfrac{0.37 \cdot 0.63}{200}}$ $\mathbf{0.37 \pm 0.067}$ **(0.303 , 0.437)**

b) Find the minimum sample size required for estimating the overall proportion of students who passed Exam P in 2017 to within 4% with 95% confidence if it is known that this proportion is between 30% and 40%.

Use $p^* = 0.40$ (closest to 0.50 possible value of p). $\varepsilon = 0.04.$

95% confidence level, $z_{\alpha/2} = z_{0.025} = 1.96.$

$$n = \left(\frac{z_{\alpha/2}}{\varepsilon} \right)^2 \cdot p^* \cdot (1 - p^*) = \left(\frac{1.96}{0.04} \right)^2 \cdot 0.40 \cdot 0.60 = 576.24.$$

Round up. $n = \mathbf{577}.$

c) In 2016, the success rate on Exam P was 41%. At a 10% level of significance, test whether the success rate on Exam P in 2017 has changed from 2016. Find the p-value of this test.

$H_0 : p = 0.41$ vs. $H_1 : p \neq 0.41.$

Test Statistic: $$Z = \frac{\hat{p} - p_0}{\sqrt{\dfrac{p_0 \cdot (1 - p_0)}{n}}} = \frac{0.37 - 0.41}{\sqrt{\dfrac{0.41 \cdot 0.59}{200}}} = \mathbf{-1.15}.$$

P-value: Two – tailed.

P-value = $2 \times P(Z \leq -1.15) = 2 \times 0.1251 = \mathbf{0.2502}$.

Decision:

P-value > $\alpha = 0.10$. **Do NOT Reject H_0 at $\alpha = 0.10$.**

1. (continued)

Suppose also that in a random sample of 150 students who took Exam FM in 2017, 45 successfully passed the exam. (Exam FM [Financial Mathematics] is the second Actuarial Science exam.)

d) Construct a 95% confidence interval for the difference between the overall proportions of students who passed Exam P and Exam FM in 2017.

$\hat{p}_{P} = 0.37.$ $\hat{p}_{FM} = \frac{45}{150} = 0.30.$

$$\left(\hat{p}_{P} - \hat{p}_{FM}\right) \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_{P} \cdot \left(1 - \hat{p}_{P}\right)}{n_{P}} + \frac{\hat{p}_{FM} \cdot \left(1 - \hat{p}_{FM}\right)}{n_{FM}}}$$

95% confidence level, $z_{\alpha/2} = z_{0.025} = 1.96.$

$$\left(0.37 - 0.30\right) \pm 1.96 \cdot \sqrt{\frac{0.37 \cdot 0.63}{200} + \frac{0.30 \cdot 0.70}{150}}$$

$\mathbf{0.070 \pm 0.099}$ $\mathbf{(-0.029, 0.169)}$

e) At a 10% level of significance, test whether Exam P had a higher success rate than Exam FM in 2017. Find the p-value of this test.

$H_0: p_{P} = p_{FM}$ vs. $H_1: p_{P} > p_{FM}.$ $\hat{p} = \frac{74 + 45}{200 + 150} = \frac{119}{350} = 0.34.$

Test Statistic: $$Z = \frac{0.37 - 0.30}{\sqrt{0.34 \cdot 0.66 \cdot \left(\frac{1}{200} + \frac{1}{150}\right)}} = \mathbf{1.37}.$$

P-value: Right – tailed.

P-value = (area of the right tail) = $P(Z \geq 1.37) = \mathbf{0.0853}$.

3. A random sample of 9 adult white rhinos had the sample mean weight of 5,100 pounds and the sample standard deviation of 450 pounds. A random sample of 16 adult hippos had the sample mean weight of 3,300 pounds and the sample standard deviation of 400 pounds. Assume that the two populations are approximately normally distributed.

a) Construct a 95% confidence interval for the difference between their overall average weights of adult white rhinos and adult hippos.

i) Assume that the overall standard deviations are equal.

$$s_{\text{pooled}}^2 = \frac{(9-1)\cdot 450^2 + (16-1)\cdot 400^2}{9+16-2} \approx 174{,}782.6 \qquad s_{\text{pooled}} \approx 418.07$$

$$\left(\overline{\mathrm{X}} - \overline{\mathrm{Y}}\right) \pm t_{\alpha/2} \cdot s_{\text{pooled}} \cdot \sqrt{\frac{1}{n} + \frac{1}{m}} \qquad 9 + 16 - 2 = \mathbf{23} \text{ degrees of freedom}$$

$$t_{0.025}(23) = 2.069 \qquad (5{,}100 - 3{,}300) \pm 2.069 \cdot 418.07 \cdot \sqrt{\frac{1}{9} + \frac{1}{16}}$$

1,800 ± 360.4 **(1,439.6, 2,160.4)**

ii) Do NOT assume that the overall standard deviations are equal. Use Welch's T.

$$\left\lfloor \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1}\cdot\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\cdot\left(\frac{s_2^2}{n_2}\right)^2} \right\rfloor = \left\lfloor \frac{\left(\frac{450^2}{9} + \frac{400^2}{16}\right)^2}{\frac{1}{9-1}\cdot\left(\frac{450^2}{9}\right)^2 + \frac{1}{16-1}\cdot\left(\frac{400^2}{16}\right)^2} \right\rfloor$$

$$= \lfloor 15.1 \rfloor = \mathbf{15} \text{ degrees of freedom}$$

$$t_{0.025}(15) = 2.131 \qquad (5{,}100 - 3{,}300) \pm 2.131 \cdot \sqrt{\frac{450^2}{9} + \frac{400^2}{16}}$$

1,800 ± 384.17 **(1,415.83, 2,184.17)**

iii) Do NOT assume that the overall standard deviations are equal. Use conservative approach.

$t_{0.025}(8) = 2.306$ $\quad\quad$ $(5{,}100 - 3{,}300) \pm 2.306 \cdot \sqrt{\dfrac{450^2}{9} + \dfrac{400^2}{16}}$

1,800 ± 415.72 $\quad\quad$ **(1,384.28 , 2,215.72)**

b) Test $H_0 : \mu_R - \mu_H = 1{,}500$ vs. $H_1 : \mu_R - \mu_H > 1{,}500$ at a = 0.05.

i) Assume that the overall standard deviations are equal.

Test Statistic: $$T = \frac{(\overline{X} - \overline{Y}) - 1{,}500}{s_{\text{pooled}} \cdot \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{(5{,}100 - 3{,}300) - 1{,}500}{418.07 \cdot \sqrt{\dfrac{1}{9} + \dfrac{1}{16}}} \approx 1.722.$$

Rejection Region: $T > t_{0.05}(23 \text{ degrees of freedom}) = 1.714.$

Decision: **Reject $\mathbf{H_0}$** at $\alpha = 0.05$.

p-value ≈ 0.0492.

ii) Do NOT assume that the overall standard deviations are equal. Use Welch's T.

Test Statistic: $$T = \frac{(\overline{X} - \overline{Y}) - 1{,}500}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \frac{(5{,}100 - 3{,}300) - 1{,}500}{\sqrt{\dfrac{450^2}{9} + \dfrac{400^2}{16}}} \approx 1.664.$$

Rejection Region: $T > t_{0.05}(15 \text{ degrees of freedom}) = 1.753.$

Decision: **Do NOT Reject $\mathbf{H_0}$** at $\alpha = 0.05$.

p-value ≈ 0.0584.

4. In a random sample of eight 8-to-12-year-old children, the sample mean time playing video games was 48.8 hours per month with sample standard deviation 3.5 hours. In a random sample of ten 13-to-18-year-olds, the sample mean time playing video games was 54.3 hours per month with sample standard deviation 5.3 hours. Assume that the two populations are normally distributed.

a) Assume that the two population variances are equal. Construct a 95% confidence interval for the difference between are average time 8-to-12-year-olds and 13-to-18-year-olds play video games per month.

$$\left(\overline{\mathrm{X}} - \overline{\mathrm{Y}}\right) \pm t_{\alpha/2} \cdot s_{\text{pooled}} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$s_{\text{pooled}}^{2} = \frac{(8-1)\cdot 3.5^2 + (10-1)\cdot 5.3^2}{8+10-2} = 21.16 \qquad s_{\text{pooled}} = 4.6$$

8 + 10 – 2 = 16 degrees of freedom

$t_{0.025}(16) = 2.120$

$$(54.3 - 48.8) \pm 2.120 \cdot 4.6 \cdot \sqrt{\frac{1}{8} + \frac{1}{10}}$$

5.5 ± 4.626 $\qquad$ **(0.874, 10.126)**

b) Since the larger sample variance is more than twice as big as the smaller one, the assumption of equal variances is questionable here. Construct a 95% confidence interval for the difference between are average time 8-to-12-year-olds and 13-to-18-year-olds play video games per month without assuming that the two population variances are equal. Use Welch's T.

$$\left(\overline{\mathrm{X}}-\overline{\mathrm{Y}}\right) \pm t_{\alpha/2} \cdot \sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}$$

$$\left\lfloor \frac{\left(\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\cdot\left(\frac{s_1^2}{n_1}\right)^2+\frac{1}{n_2-1}\cdot\left(\frac{s_2^2}{n_2}\right)^2} \right\rfloor = \left\lfloor \frac{\left(\frac{3.5^2}{8}+\frac{5.3^2}{10}\right)^2}{\frac{1}{8-1}\cdot\left(\frac{3.5^2}{8}\right)^2+\frac{1}{10-1}\cdot\left(\frac{5.3^2}{10}\right)^2} \right\rfloor$$

$$= \lfloor 15.5468 \rfloor = 15 \text{ degrees of freedom}$$

$$t_{0.025}(15) = 2.131$$

$$(54.3-48.8) \pm 2.131 \cdot \sqrt{\frac{3.5^2}{8}+\frac{5.3^2}{10}}$$

$\mathbf{5.5 \pm 4.44}$ $\qquad$ $\mathbf{(1.06, 9.94)}$

5. A researcher wishes to determine whether the starting salaries of high-school math teachers in private schools are higher than those of high-school math teachers in public schools. She selects a sample of new math teachers from each type of school and calculates the sample means and sample standard deviations of their salaries.

	Private	Public
sample size	10	7
sample mean	$36,800	$36,300
sample standard deviation	$600	$546

Assume that the populations are normally distributed and the population variances are equal.

a) Construct a 95% confidence interval for the difference in average starting salaries of high-school math teachers in private and public schools.

$$s_{\text{pooled}}^2 = \frac{(10-1)\cdot 600^2 + (7-1)\cdot 546^2}{10+7-2} = 335246.4 \qquad s_{\text{pooled}} \approx 579$$

$$\left(\overline{X}-\overline{Y}\right) \pm t_{\alpha/2}\cdot s_{\text{pooled}}\cdot\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

$10 + 7 - 2 = 15$ degrees of freedom

$\alpha = 0.05 \qquad t_{0.025}(15) = 2.131$

$$(36{,}800 - 36{,}300) \pm 2.131\cdot 579\cdot\sqrt{\frac{1}{10}+\frac{1}{7}}$$

$\mathbf{500 \pm 608} \qquad \mathbf{(-108,\ 1108)}$

b) Perform the appropriate test at a 1% level of significance. That is, test

$H_0: \mu_{\text{Private}} = \mu_{\text{Public}}$ vs. $H_1: \mu_{\text{Private}} > \mu_{\text{Public}}$.

Test Statistic:
$$T = \frac{\left(\overline{X}-\overline{Y}\right) - 0}{s_{\text{pooled}}\cdot\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{(36{,}800-36{,}300)-0}{579\cdot\sqrt{\frac{1}{10}+\frac{1}{7}}} \approx 1.752.$$

Rejection Region: $T > t_{0.01}(10 + 7 - 2 = 15 \text{ degrees of freedom}) = 2.602.$

Decision: **Do NOT Reject H_0** at $\alpha = 0.01$.

c) Find the p-value of the test in part (b).

$t_{0.05}(15) = 1.753.$ $\Rightarrow$ one tail ≈ 0.05.

p-value = right tail $\approx$ **0.05**.

d) Suppose instead we wish to test $H_0: \mu_{\text{Private}} = \mu_{\text{Public}}$ vs. $H_1: \mu_{\text{Private}} \neq \mu_{\text{Public}}$.
(We wish to determine whether the starting salaries of high-school math teachers in private schools are different from those of high-school math teachers in public schools.)
What is the p-value for this test?

p-value = two tails $\approx$ **0.10**.

6. Six children are tested for pulse rate before and after watching a violent movie with the following results.

Child	1	2	3	4	5	6
Before	102	96	89	104	90	85
After	109	105	94	112	104	96

Using the paired t test, test for differences in the before and after mean pulse rates. Use $\alpha = 0.05$ and use a two-sided test.

Before	102	96	89	104	90	85
After	109	105	94	112	104	96
Difference	7	9	5	8	14	11

$$\bar{d} = \frac{7+9+5+8+14+11}{6} = \frac{54}{6} = \mathbf{9}.$$

$\sum d^2 = 49 + 81 + 25 + 64 + 196 + 121 = 536.$

$$s_d^2 = \frac{\sum d^2 - n \cdot \bar{d}^2}{n-1} = \frac{536 - 6 \cdot 9^2}{6-1} = \frac{50}{5} = \mathbf{10}.$$

OR

$$s_d^2 = \frac{\sum \left(d - \bar{d}\right)^2}{n-1} = \frac{4+0+16+1+25+4}{5} = \frac{50}{5} = \mathbf{10}.$$

$H_0:\ \mu_B = \mu_A$ vs. $H_1:\ \mu_B \neq \mu_A$. $\Leftrightarrow$ $H_0:\ \mu_D = 0$ vs. $H_1:\ \mu_D \neq 0$.

Test Statistic: $$T = \frac{\bar{d} - 0}{s_d \big/ \sqrt{n}} = \frac{9-0}{\sqrt{10} \big/ \sqrt{6}} \approx \mathbf{6.971}.$$

$n - 1 = 5$ degrees of freedom.

$\alpha = 0.05, \qquad {}^{\alpha}\!/_{2} = 0.025, \qquad \pm t_{\alpha/2} = \pm 2.571. \qquad \leftarrow$ Critical Values

The test statistic **is** in the Rejection Region. **Reject H_0 at $\alpha = 0.05$.**

OR

$n - 1 = 5$ degrees of freedom. $\qquad t_{0.005} = 4.032.$

$\Rightarrow$ Right tail is less than 0.005.

$\Rightarrow$ P-value = (two tails) is less than 0.01.

(p-value ≈ 0.000934)

P-value $< 0.05 = \alpha$. **Reject H_0 at $\alpha = 0.05$.**

OR

Confidence interval: $\bar{d} \pm t_{\alpha/2} \cdot \dfrac{s_d}{\sqrt{n}}.$

$n - 1 = 5$ degrees of freedom.

$\alpha = 0.05, \qquad {}^{\alpha}\!/_{2} = 0.025, \qquad t_{\alpha/2} = 2.571.$

$9 \pm 2.571 \cdot \dfrac{\sqrt{10}}{\sqrt{6}} \qquad \mathbf{9 \pm 3.32} \qquad \mathbf{(5.68,\ 12.32)}$

95% confidence interval does NOT cover zero $\Leftrightarrow$ **Reject H_0 at $\alpha = 0.05$.**