STAT 400: Homework 01

Spring 2018, UIUC

Due: Friday, January 26, 2:00 PM

Exercise 1

(a) Evaluate the following integral. Do **not** use a calculator or computer, except to check your work.

$$\int_0^\infty xe^{-2x}dx$$

Solution:

Here we have integration by parts. We set

$$u = x, \quad dv = e^{-2x}dx$$

Thus we have

$$du = dx, \quad v = -\frac{1}{2}e^{-2x}$$

Then we obtain

$$\int_0^\infty xe^{-2x}dx = -\frac{1}{2}xe^{-2x}\bigg|_0^\infty + \int_0^\infty \frac{1}{2}e^{-2x}dx = \boxed{\frac{1}{4}} = \boxed{0.25}$$

Note that we are being somewhat abusive with notation since we are dealing with an improper intergral.

(b) Evaluate the following integral. Do **not** use a calculator or computer, except to check your work.

$$\int_0^\infty xe^{-x^2}dx$$

Solution:

Here we use integration by substitution. We set

$$u = x^2$$

Thus we have

$$du = 2xdx$$

Then we obtain

$$\int_0^\infty xe^{-x^2}dx = \int_0^\infty \frac{1}{2}e^{-x^2}(2xdx) = \frac{1}{2}\int_0^\infty e^{-u}du = -\frac{1}{2}e^{-u}\bigg|_0^\infty = \boxed{\frac{1}{2}} = \boxed{0.50}$$

Exercise 2

Find the value c such that

$$\iint_A cx^2y^3dydx = 1$$

where $A = \{(x, y) : 0 < x < 1,\ 0 < y < \sqrt{x}\}$. Do **not** use a calculator or computer, except to check your work.

Solution:

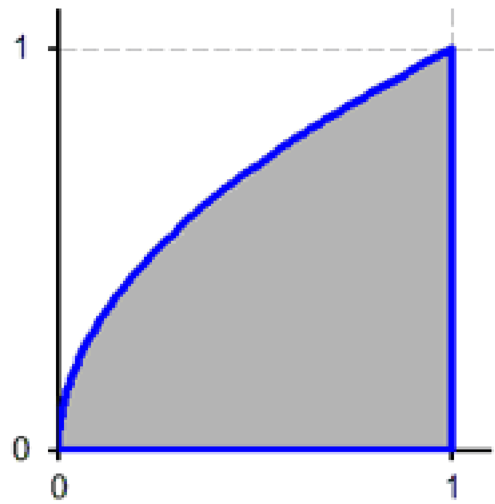

Figure 1: Integral Region

First,

$$\iint_A cx^2y^3dydx = 1 = \int_0^1 \left(\int_0^{\sqrt{x}} cx^2y^3dy\right) dx = \int_0^1 \frac{c}{4}x^4dx = \frac{c}{20}$$

Then,

$$\frac{c}{20} = 1 \implies c = \boxed{20}$$

Exercise 3

Suppose $S = \{2, 3, 4, 5, \ldots\}$ and

$$P(k) = c \cdot \frac{2^k}{k!}, \quad k = 2, 3, 4, 5, \ldots$$

Find the value of c that makes this a valid probability distribution.

Solution:

First note that,

$$\begin{aligned}\sum_{\text{all } x} P(x) &= \sum_{k=2}^{\infty} c \cdot \frac{2^k}{k!} \\ &= c \cdot \left(\sum_{k=0}^{\infty} \frac{2^k}{k!} - \frac{2^0}{0!} - \frac{2^1}{1!} \right) \\ &= c \cdot (e^2 - 1 - 2) \\ &= c \cdot (e^2 - 3)\end{aligned}$$

Then since we need to have

$$\sum_{\text{all } x} P(x) = 1$$

we obtain

$$c \cdot (e^2 - 3) = 1 \implies c = \boxed{\frac{1}{e^2 - 3}} \approx \boxed{0.22784}.$$

Exercise 4

Suppose $S = \{2, 3, 4, 5, \ldots\}$ and

$$P(k) = \frac{6}{3^k}, \quad k = 2, 3, 4, 5, \ldots$$

Find $P(\text{outcome is greater than 3})$.

Solution:

$$\begin{aligned}P(\text{outcome is greater than 3}) &= P(4) + P(5) + P(6) + \ldots \\ &= \sum_{k=4}^{\infty} \frac{6}{3^k} = \frac{\text{first term}}{1 - \text{base}} = \frac{\frac{6}{3^4}}{1 - \frac{1}{3}} \\ &= \boxed{\frac{1}{9}} = \boxed{0.111\bar{1}}\end{aligned}$$

Or alternatively,

$$\begin{aligned}P(\text{outcome is greater than 3}) &= 1 - P(2) - P(3) \\ &= 1 - \frac{6}{3^2} - \frac{6}{3^3} \\ &= \boxed{\frac{1}{9}} = \boxed{0.111\bar{1}}\end{aligned}$$

Exercise 5

Suppose $P(A) = 0.4$, $P(B') = 0.3$, and $P(A \cap B') = 0.1$.

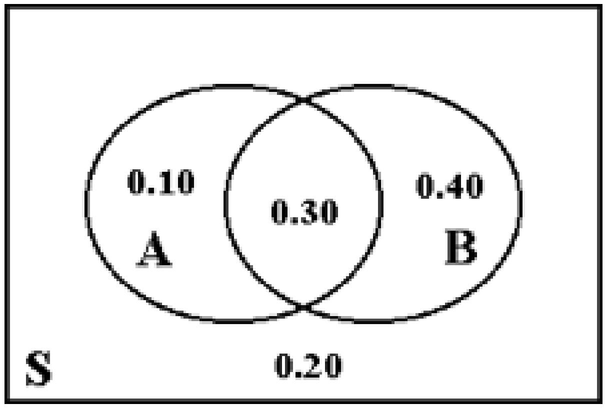

Figure 2: Venn Diagram for $P(A) = 0.4$, $P(B') = 0.3$, and $P(A \cap B') = 0.1$

	B	B'	
A	0.30	**0.10**	**0.40**
A'	0.40	0.20	0.60
	0.70	**0.30**	1.00

(a) Find $P(A \cup B)$.

Solution:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.40 + 0.70 - 0.30 = \boxed{0.80}$$

$$P(A \cup B) = 1 - P(A' \cap B') = 1 - 0.20 = \boxed{0.80}$$

$$P(A \cup B) = P(A \cap B) + P(A \cap B') + P(A' \cap B) = 0.30 + 0.10 + 0.40 = \boxed{0.80}$$

(b) Find $P(B' \mid A)$.

Solution:

$$P(B' \mid A) = \frac{P(A \cap B')}{P(A)} = \frac{0.10}{0.40} = \boxed{\frac{1}{4}} = \boxed{0.25}$$

(c) Find $P(B \mid A')$.

Solution:

$$P(B \mid A') = \frac{P(A' \cap B)}{P(A')} = \frac{0.40}{0.60} = \boxed{\frac{2}{3}} = \boxed{0.66\overline{6}}$$

Exercise 6

Suppose:

- $P(A) = 0.6$
- $P(B) = 0.5$
- $P(C) = 0.4$
- $P(A \cap B) = 0.3$
- $P(A \cap C) = 0.2$
- $P(B \cap C) = 0.2$
- $P(A \cap B \cap C) = 0.1$

(a) Find $P((A \cup B) \cap C')$.

Solution:

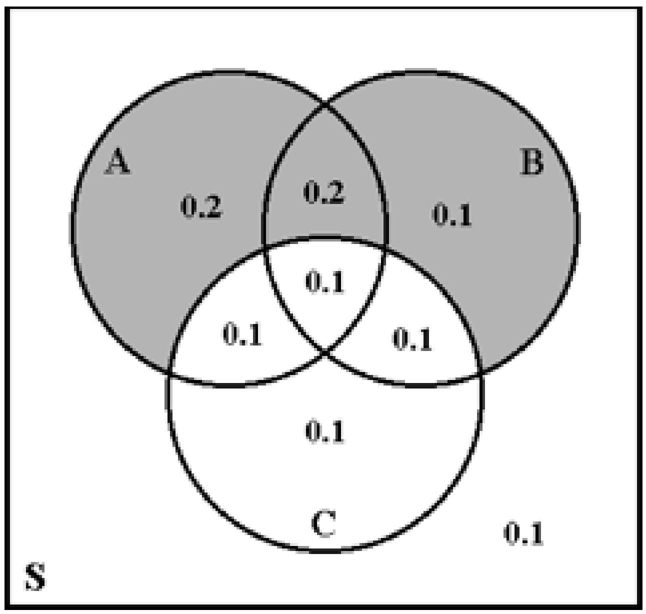

Figure 3: Venn Diagram with $P((A \cup B) \cap C')$ shaded.

After finding the probabilities for each disjoint region and shading the appropriate venn diagram, we have

$$P((A \cup B) \cap C') = \boxed{0.50}$$

(b) Find $P(A \cup (B \cap C))$.

Solution:

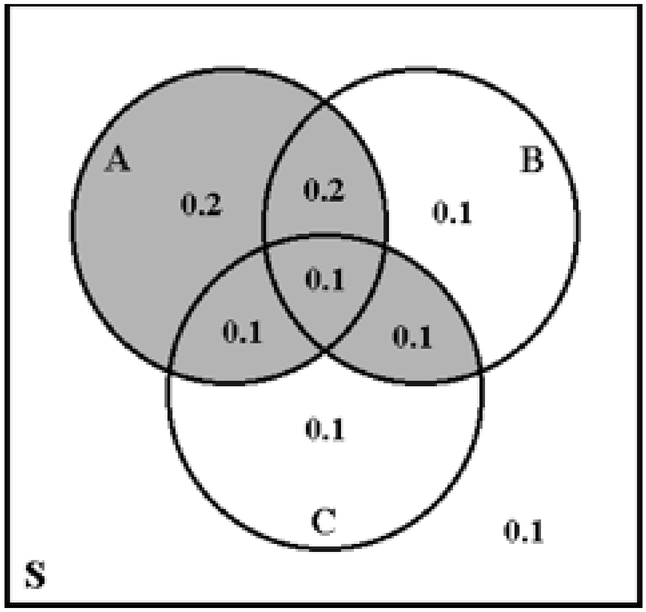

Figure 4: Venn Diagram with $P(A \cup (B \cap C))$ shaded.

After finding the probabilities for each disjoint region and shading the appropriate venn diagram, we have

$$P(A \cup (B \cap C)) = \boxed{0.70}$$